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Universality for Orthogonal and Symplectic Laguerre-Type Ensembles

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We give a proof of the Universality Conjecture for orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) random matrix ensembles of Laguerre-type in the bulk of the spectrum as well as at the hard and soft spectral edges. Our results are stated precisely in the Introduction (Theorems 1.1, 1.4, 1.6 and Corollaries 1.2, 1.5, 1.7). They concern the appropriately rescaled kernels $K_{n,\beta}$, correlation and cluster functions, gap probabilities and the distributions of the largest and smallest eigenvalues. Corresponding results for unitary ($\beta = 2$) Laguerre-type ensembles have been proved by the fourth author in Ref. 23. The varying weight case at the hard spectral edge was analyzed in Ref. 13 for $\beta = 2$: In this paper we do not consider varying weights.
Our proof follows closely the work of the first two authors who showed in Refs. 7, 8 analogous results for Hermite-type ensembles. As in Refs. 7, 8 we use the version of the orthogonal polynomial method presented in Refs. 22, 25, to analyze the local eigenvalue statistics. The necessary asymptotic information on the Laguerre-type orthogonal polynomials is taken from Ref. 23.

KEY WORDS: random matrix theory, universality, orthogonal and symplectic ensembles, Laguerre-type weights, hard edge, soft edge, bulk.

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1. INTRODUCTION

In this paper we consider ensembles of matrices $\{M\}$ with invariant distributions of Laguerre type

$$d\mathbb{P}_{n,\beta}(M) = \mathcal{P}_{n,\beta}(M)\,dM = \frac{1}{\mathcal{Z}_{n,\beta}} \det(W_\gamma(M))e^{-\operatorname{tr} Q(M)}\,dM, \tag{1.1}$$

for $\beta = 1, 2$ and 4, the so-called Orthogonal, Unitary and Symplectic ensembles, respectively (see Ref. 14). For $\beta = 1, 2, 4$, the ensemble consists of $n \times n$ real symmetric matrices, $n \times n$ Hermitian matrices, and $2n \times 2n$ Hermitian self-dual matrices (see Ref. 14), respectively. The above terminology for $\beta = 1, 2$ and 4 reflects the fact that (1.1) is invariant under conjugation of M, $M \mapsto UMU^{-1}$, by orthogonal, unitary and unitary-symplectic matrices U. Furthermore, in (1.1), dM denotes Lebesgue measure on the algebraically independent entries of M, $W_\gamma(x) = x^\gamma \mathbf{1}_{\mathbb{R}_+}(x)$ with $\gamma > 0$, Q denotes any polynomial of positive degree and with positive leading coefficient, and $\mathcal{Z}_{n,\beta}$ is a normalization constant. Of course, $\mathcal{P}_{n,\beta}$ and $\mathcal{Z}_{n,\beta}$ depend not only on n and β which are implicit in (1.1) but also on the quantities γ and Q. For the sake of readability the dependence on γ and Q is suppressed in all of our notation.

For ensembles (1.1) the joint probability density function for the eigenvalues $x_1, x_2, \ldots, x_n$ of M is given by (see Ref. 14)

$$P_{n,\beta}(x_1, \ldots, x_n) = \frac{1}{Z_{n,\beta}} \prod_{1\le j<k\le n} |x_j - x_k|^\beta \prod_{j=1}^{n} w_\beta(x_j) \quad \text{on } \mathbb{R}_+^n \tag{1.2}$$

where again $Z_{n,\beta}$ denotes the corresponding normalization constant and

$$w_\beta(x) = \begin{cases} x^\gamma e^{-Q(x)}, & \beta = 1, 2 \\ \left(x^\gamma e^{-Q(x)}\right)^2, & \beta = 4. \end{cases} \tag{1.3}$$

The second power appearing in $w_{\beta=4}$ simply reflects the fact that the eigenvalues of self-dual Hermitian matrices come in pairs.

Our main results stated below show that the appropriately rescaled local eigenvalue statistics for ensembles (1.1) are universal (i.e. independent of Q) in the limit $n \to \infty$, where for $\beta = 1$ only matrices of even dimension are considered.[5] Consequently, the limiting local eigenvalue statistics agree for all ensembles (1.1) with the corresponding limiting statistics in the well studied classical cases of linear Q (see e.g. Refs. 10, 11, 16, 17, 21 and references therein). Ensembles (1.1) with linear Q are called Laguerre ensembles because w_β in (1.3) is then a

[5] For matrices of odd dimension in the case $\beta = 1$, see the discussion following Eq. (1.13) in Ref. 7.

Laguerre weight. More generally, all matrix ensembles with eigenvalue probability density function of the form (1.2), (1.3) and with linear Q are called Laguerre ensembles irrespective of whether they arise from matrix ensembles of the form (1.1). In fact, Laguerre ensembles appeared first in statistics and in physics and these were not of type (1.1). In statistics, for example, Wishart ensembles $\{M\}$ with $M = X^t X$ and X being a random $N \times n\,(N \geq n)$ rectangular matrix with real entries that are independently distributed standard Gaussian variables, have an eigenvalue probability density function of the form (1.2), (1.3) with $\beta = 1$, $\gamma = (N - n - 1)/2$ and $Q(x) = x/2$ (see e.g. Ref. 15). In physics, Laguerre ensembles emerge e.g. in the study of Dirac operators in quantum chromodynamics and in the study of disordered superconductors in mesoscopic physics, see e.g. Refs. 4, 24. Here we encounter not only Wishart ensembles but also random matrices with a 2×2 block structure which lead again to an eigenvalue probability density function of the form (1.2), (1.3). For example, random Dirac operators in the chiral gauge are modelled by $\left(\begin{smallmatrix} 0 & X \\ X^t & 0 \end{smallmatrix}\right)$ where X is a rectangular $N \times n$ random matrix. Choosing again the entries of X to be independently distributed real standard Gaussian variables one obtains a density function for (the squares of) the eigenvalues which is of the form (1.2), (1.3) with $\beta = 1$, $\gamma = (N - n - 1)/2$ and $Q(x) = x/2$.

In Refs. 7, 8 the authors proved universality in the bulk[7] and at the spectral edge[8] for *Hermite-type ensembles,* i.e. for ensembles (1.1) with $W_\gamma(x) = 1$ for all $x \in \mathbb{R}$ and with $Q(x)$ denoting any polynomial of even positive degree and with positive leading coefficient. To the best of our knowledge, universality results for Laguerre-type ensembles have so far only been proved for unitary ($\beta = 2$) ensembles in Ref. 13 (varying weights) and in Ref. 23 where the author showed universality for unitary ensembles of the form (1.1). All the results regarding $\beta = 2$ stated in the present paper can be found already in Ref. 23 and we only include them here for the sake of completeness. Moreover, a number of formulae and estimates proved in Ref. 23 play a key role in our proof of universality for $\beta = 1, 4$. Universality for Laguerre-type ensembles, for all three cases $\beta = 1$, 2 and 4, has been considered in the physics literature (see e.g. Refs. 3, 18 and references therein). More information on the history of universality for matrix ensembles can be found the introductions of Refs. 7, 8 and in Ref. 6.

The basic structure of the proof in this paper is similar to Refs. 7, 8 and relies on the orthogonal polynomial method developed in Refs. 22 and 25. A detailed description of the strategy of proof can be found in the Introductions of Refs. 7 and 8. We now introduce some further notation that is needed to state our main results.

Following,[25] (Ref. 7, Remark 1.3) we define weights of the form

$$w(x) = x^\alpha e^{-V(x)}, \quad \text{for } x \in \mathbb{R}_+, \tag{1.4}$$

with

$$\alpha := \begin{cases} \gamma, & \beta = 2 \\ 2\gamma, & \beta = 1, 4 \end{cases}; \qquad V := \begin{cases} Q, & \beta = 2 \\ 2Q, & \beta = 1, 4 \end{cases} \tag{1.5}$$

(γ, Q as in (1.1)) in order to be able to use the same set of orthogonal polynomials in all three cases $\beta = 1, 2, 4$. By the assumptions made on γ and Q we will assume that

$$\alpha > 0 \quad \text{and} \quad V(x) = \sum_{j=0}^{m} q_j x^j \tag{1.6}$$

where the polynomial V, known as the external field, has positive degree m and positive leading coefficient q_m. The orthogonal polynomials p_k with respect to the weight w are uniquely defined by the conditions

$$\int_0^\infty p_k(x) p_l(x) w(x)\, dx = \delta_{k,l} \quad \text{for } k, l \in \mathbb{N}_0,$$

and $p_k(x) = \gamma_k x^k + \cdots$ is a polynomial of degree k with positive leading coefficient $\gamma_k > 0$. The functions

$$\phi_k(x) := p_k(x)\sqrt{w(x)} \tag{1.7}$$

then form an orthonormal system in $L_2(\mathbb{R}_+)$. The statement of our main results involves several quantities that arise in the asymptotic analysis of the orthogonal polynomials p_k, viz., the Mhaskar–Rakhmanov–Saff numbers β_n, the densities ω_n of the equilibrium measures in the presence of the rescaled external field $V_n(x) = \frac{1}{n} V(\beta_n x)$, and numbers c_n, $\tilde{c}_n$ related to the behavior of the equilibrium measure at the soft, hard edges respectively. The definition and relevant properties of all these quantities are summarized in Eqs. (4.3)–(4.12) of Sec. 4.1 below where one can also find references to Ref. 23 for their respective derivations.

As mentioned above our proof relies on the orthogonal polynomial method for invariant matrix ensembles. This method is based on the observation that the eigenvalue statistics (e.g. correlation and cluster functions, gap probabilities, distributions of smallest and largest eigenvalues) can be analyzed using functions $K_{n,\beta}$ of two variables which can be expressed in terms of the orthogonal polynomials p_k (see Ref. 22). More precisely, let ε denote the integral operator with kernel $\varepsilon(x, y) = \frac{1}{2}\operatorname{sgn}(x - y)$ where $\operatorname{sgn} = \mathbf{1}_{\mathbb{R}_+} - \mathbf{1}_{\mathbb{R}_-}$ is the standard sign-function. We then define

$$K_{n,2}(x,y) := K_n(x,y) := \sum_{k=0}^{n-1} \phi_k(x)\phi_k(y) \quad \text{(Christoffel–Darboux kernel)} \tag{1.8}$$

$$K_{n,1}(x,y) = \begin{pmatrix} S_{n,1}(x,y) & -\frac{\partial}{\partial y} S_{n,1}(x,y) \\ (\varepsilon S_{n,1})(x,y) - \frac{1}{2}\operatorname{sgn}(x-y) & S_{n,1}(y,x) \end{pmatrix}, \quad \text{for } n \text{ even}^5, \tag{1.9}$$

$$K_{n,4}(x,y) = \frac{1}{2}\begin{pmatrix} S_{n,4}(x,y) & -\frac{\partial}{\partial y} S_{n,4}(x,y) \\ (\varepsilon S_{n,4})(x,y) & S_{n,4}(y,x) \end{pmatrix}. \tag{1.10}$$

Here $S_{n,\beta}$ ($\beta = 1, 4$) are certain specific scalar functions which will be discussed in detail in Sec. 2. The analysis in the present paper depends critically on the formulae of Widom (Ref. 25, Theorem 2) that express the functions $S_{n,\beta}$ in terms of the orthogonal polynomials p_k.

We will prove the convergence of $K_{n,\beta}$ for $n \to \infty$ to a universal limit that is independent of V. In proving the convergence one needs to rescale the arguments x and y appropriately. Since the (1,2)-entry of $K_{n,\beta}$ for $\beta = 1$, 4 contains differentiation with respect to y, and the (2,1)-entry of $K_{n,\beta}$ contains integration with respect to x, these two entries behave differently under rescaling. In order to take this into account it is convenient to introduce the following notation for $\beta = 1, 4$:

$$K_{n,\beta}^{(\lambda)} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} K_{n,\beta} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} (K_{n,\beta})_{11} & \lambda^{-2}(K_{n,\beta})_{12} \\ \lambda^2 (K_{n,\beta})_{21} & (K_{n,\beta})_{22} \end{pmatrix}, \quad \lambda > 0. \tag{1.11}$$

We now are ready to state our main results. Since the statistical behavior is different for eigenvalues in the bulk of the spectrum and at the spectral edges, we need to distinguish these cases. Moreover, for Laguerre-type ensembles the lower and upper spectral edges have a different character. The lower edge at the origin is called a *hard edge*, because no eigenvalue can be less than zero by definition of the ensemble. For the upper edge, on the other hand, there is no apriori upper bound for the eigenvalues. The existence of the upper spectral edge is due to the fact that the probability for an eigenvalue to be bigger than a certain n-dependent threshold value is exponentially small: This threshold value is known as the *soft edge* of the spectrum. Both the rescaling and the limit of $K_{n,\beta}$ are different for the bulk, for the soft edge and for the hard edge. In Refs. 7, 8 the authors proved universality for Hermite-type ensembles in the bulk and at the soft edge, respectively. We state the analogous results for Laguerre-type ensembles in Theorems 1.6, 1.4 below. Note that another manifestation of universality is seen in the fact that the limits of the appropriately rescaled $K_{n,\beta}$ are the same for Hermite-type and Laguerre-type ensembles both in the bulk and at the soft edge.

We start by stating our results for the *hard edge*, a case which is not present in Hermite-type ensembles.[7,8]

Notational remark. In Theorem 1.1 and also in other situations where we consider the hard edge, we will use the notation that an estimate holds *uniformly for* ξ, η *in bounded subsets of* $(0, \infty)$. By this we mean that the estimate holds for ξ, η in any set of the form $(0, L)$, $0 < L < \infty$. By *uniformly* we mean that the constant in the $\mathcal{O}$-term in (1.13) below, for example, depends only on L. This somewhat unusual notation is necessitated by the actual form of the error estimates for the correlation kernel near 0, see e.g. (1.13) and the proof of Corollary 1.2(b) in Sec. 6.1 below.

Notational comment. In order to treat the orthogonal and symplectic ensembles simultaneously (see discussion following (1.5) above and also the discussion in the paragraph preceding (2.8) below) we consider ensembles of matrices of size n, n even[5], for $\beta = 1$ and 4. In the light of definition (1.1), for $\beta = 4$, n should be interpreted as $n = 2 \cdot \frac{n}{2}$.

Theorem 1.1. (hard edge) *Let* $\beta = 1, 2$ *or* 4 *and introduce the notation*

$$\nu_n = \left(\frac{\beta_n}{4\tilde{c}_n n^2}\right)^{-1/2}, \qquad \tilde{x}^{(n)} = \frac{1}{\nu_n^2} x = \frac{\beta_n}{4\tilde{c}_n n^2} x.$$

Then, as $n \to \infty$ *(*n *even for the cases* $\beta = 1, 4$*) the following holds uniformly for* ξ, η *in bounded subsets of* $(0, \infty)$.

(i) *The case* $\beta = 2$*:*

$$\frac{1}{\nu_n^2} K_n\left(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\right) = K_J(\xi, \eta) + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}} \eta^{\frac{\alpha}{2}}}{n}\right). \tag{1.12}$$

where K_J *denotes the Bessel kernel,*

$$K_J(\xi, \eta) = \frac{J_\alpha(\sqrt{\xi})\sqrt{\eta} J'_\alpha(\sqrt{\eta}) - J_\alpha(\sqrt{\eta})\sqrt{\xi} J'_\alpha(\sqrt{\xi})}{2(\xi - \eta)}.$$

(ii) *The case* $\beta = 4$*:*

$$\frac{1}{\nu_n^2} K^{(\nu_n)}_{\frac{n}{2},4}\left(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\right) = K^{(4)}(\xi, \eta) + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}} \eta^{\frac{\alpha}{2}}}{n}\right) \begin{pmatrix} \xi^{-1} & \xi^{-1}\eta^{-1} \\ 1 & \eta^{-1} \end{pmatrix}, \tag{1.13}$$

where

$$\begin{aligned}
2\big(K^{(4)}\big)_{11}(\xi,\eta) &= 2\big(K^{(4)}\big)_{22}(\eta,\xi)\\
&= K_J(\xi,\eta) + \frac{1}{4}\left(\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}} - \frac{2\alpha}{\xi}J_\alpha(\sqrt{\xi})\right)\\
&\quad\times \int_0^{\sqrt{\eta}} J_{\alpha+1}(s)\,ds,\\
2\big(K^{(4)}\big)_{12}(\xi,\eta) &= -\frac{\partial}{\partial\eta}K_J(\xi,\eta) - \frac{1}{8}\left(\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}} - \frac{2\alpha}{\xi}J_\alpha(\sqrt{\xi})\right)\\
&\quad\times \frac{J_{\alpha+1}(\sqrt{\eta})}{\sqrt{\eta}},\\
2\big(K^{(4)}\big)_{21}(\xi,\eta) &= \int_0^{\xi} K_J(s,\eta)\,ds + \frac{1}{2}\int_0^{\sqrt{\xi}}\left(J_{\alpha+1}(s) - \frac{2\alpha}{s}J_\alpha(s)\right)ds\\
&\quad\times \int_0^{\sqrt{\eta}} J_{\alpha+1}(s)\,ds.
\end{aligned}$$

(iii) *The case* $\beta = 1$*: there exists* $0 < \tau = \tau(m,\alpha) < 1$ *such that*

$$\frac{1}{\nu_n^2}K_{n,1}^{(\nu_n)}\big(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\big) = K^{(1)}(\xi,\eta) + \mathcal{O}(n^{-\tau})\begin{pmatrix}\xi^{\frac{\alpha}{2}} & \xi^{\frac{\alpha}{2}}\eta^{\frac{\alpha}{2}-1}\\ 1 & \eta^{\frac{\alpha}{2}}\end{pmatrix}, \quad (1.14)$$

where

$$\begin{aligned}
\big(K^{(1)}\big)_{11}(\xi,\eta) &= \big(K^{(1)}\big)_{22}(\eta,\xi)\\
&= K_J(\xi,\eta) - \frac{1}{4}\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}}\int_{\sqrt{\eta}}^{\infty}\left(J_{\alpha+1}(s) - \frac{2\alpha}{s}J_\alpha(s)\right)ds,\\
\big(K^{(1)}\big)_{12}(\xi,\eta) &= -\frac{\partial}{\partial\eta}K_J(\xi,\eta) - \frac{1}{8}\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}}\left(\frac{J_{\alpha+1}(\sqrt{\eta})}{\sqrt{\eta}} - \frac{2\alpha}{\eta}J_\alpha(\sqrt{\eta})\right),\\
\big(K^{(1)}\big)_{21}(\xi,\eta) &= -\int_{\xi}^{\eta} K_J(s,\eta)\,ds + \frac{1}{2}\int_{\sqrt{\xi}}^{\sqrt{\eta}} J_{\alpha+1}(s)\,ds\\
&\quad\times \int_{\sqrt{\eta}}^{\infty}\left(J_{\alpha+1}(s) - \frac{2\alpha}{s}J_\alpha(s)\right)ds - \frac{1}{2}\operatorname{sgn}(\xi-\eta).
\end{aligned}$$

As in Refs. 7, 8 we now present two consequences of Theorem 1.1 which demonstrate the relevance of the theorem for the understanding of the local eigenvalue statistics in the limit $n \to \infty$. Here we consider the distribution of the lowest eigenvalue as well as the l-point correlation functions. The latter are obtained from the probability density function $P_{n,\beta}$ essentially by integrating out the last $n-l$ variables,

$$R_{n,\beta,l}(x_1,\dots x_l) := \binom{n}{n-l} \int_{\mathbb{R}^{n-l}} P_{n,\beta}(x_1,\dots,x_n)\, dx_{l+1}\dots dx_n. \quad (1.15)$$

Corollary 1.2. *With the notation of Theorem* 1.1 *and* (1.15) *and* $\lambda_1(M)$ *denoting the smallest eigenvalue of* M *we have for* $l \in \mathbb{N}$, $\xi, \xi_i \in (0,\infty)$ *that the following limits*

(a) $$\lim_{n\to\infty} \frac{1}{\nu_n^{2l}} R_{n,\beta,l}\left(\frac{\xi_1}{\nu_n^2},\dots,\frac{\xi_l}{\nu_n^2}\right) \quad \textit{for } \beta = 1, 2;$$
$$\lim_{n\to\infty} \frac{1}{\nu_n^{2l}} R_{n/2,4,l}\left(\frac{\xi_1}{\nu_n^2},\dots,\frac{\xi_l}{\nu_n^2}\right),$$

(b) $$\lim_{n\to\infty} \mathbb{P}_{n,\beta}\left(\left\{M: \lambda_1(M) \le \frac{\xi}{\nu_n^2}\right\}\right) \quad \textit{for } \beta = 1,2;$$
$$\lim_{n\to\infty} \mathbb{P}_{\frac{n}{2},4}\left(\left\{M: \lambda_1(M) \le \frac{\xi}{\nu_n^2}\right\}\right)$$

exist (with n *even for* $\beta = 1, 4$*) and are independent of* Q *(cf.* (1.1)*).*

Existence and universality of the limits appearing in statement (a) of the Corollary follow from the convergence of the cluster functions and the relation between cluster and correlation functions (see Ref. 22, Sec. 2). The convergence of the cluster functions is immediate from Theorem 1.1 together with the formulae in (Ref. 22, Sec. 3) which express the cluster functions in terms of the kernels $K_{n,\beta}$. For $\beta = 1, 4$ one needs to observe in addition that the formulae do not change if one replaces $K_{n,\beta}$ by $K_{n,\beta}^{(\lambda)}$. The proof of existence and universality of the limits in statement (b) of the corollary is slightly more involved and will be presented at the end of Sec. 6.1.

Remark 1.3. It is also possible to give explicit formulae for the limits considered in Corollary 1.2 in terms of the kernels K_J, $K^{(1)}$ and $K^{(4)}$ for $\beta = 2, 1, 4$ respectively. These limits are easy to derive for the correlation functions (a), using the determinantal formula for $\beta = 2$ and using the relation with cluster functions for $\beta = 1, 4$.

In contrast, the dependence of the limiting distribution of the smallest eigenvalue (b) on the limiting kernels K_J, $K^{(1)}$ and $K^{(4)}$ is given via Fredholm determinants (cf. (6.29), (6.32), (6.33)) and therefore is far more complicated. However, our universality result stated in Corollary 1.2 implies that it suffices to understand the limiting distribution in the classical Laguerre case where the polynomial Q in (1.1) has degree 1. Fortunately, this case has already been studied in the literature and it was found that the limiting distributions of the smallest eigenvalue can be expressed in terms of certain Painlevé functions (see Ref. 21 for $\beta = 2$ and Ref. 11 for $\beta = 1, 4$).

Next we state our main result for the upper spectral edge.

Theorem 1.4. (soft edge) *(cf. Ref. 8, Theorem 1.1). Let $\beta = 1, 2$ or 4 and introduce the notation*

$$\lambda_n = \left(\frac{\beta_n}{c_n n^{2/3}}\right)^{-1/2}, \quad x^{(n)} = \beta_n + \frac{x}{\lambda_n^2} = \beta_n\left(1 + \frac{x}{c_n n^{2/3}}\right).$$

Fix a number L_0. Then, there exists $c = c(L_0)$ and $0 < \tau = \tau(m, \alpha) < 1$ such that as $n \to \infty$ (n even for the cases $\beta = 1, 4$) the following holds uniformly for $\xi, \eta \in [L_0, +\infty)$.

(i) *The case $\beta = 2$:*

$$\frac{1}{\lambda_n^2} K_n\big(\xi^{(n)}, \eta^{(n)}\big) = K_{\mathrm{Ai}}(\xi, \eta) + \mathcal{O}(n^{-1/3}) e^{-c\xi} e^{-c\eta}. \tag{1.16}$$

where K_{Ai} denotes the Airy kernel,

$$K_{\mathrm{Ai}}(\xi, \eta) = \frac{\mathrm{Ai}(\xi)\mathrm{Ai}'(\eta) - \mathrm{Ai}(\eta)\mathrm{Ai}'(\xi)}{\xi - \eta}.$$

(ii) *The case $\beta = 4$:*

$$\frac{1}{\lambda_n^2} K_{\frac{n}{2},4}^{(\lambda_n)}\big(\xi^{(n)}, \eta^{(n)}\big) = K^{(4)}(\xi, \eta) + \mathcal{O}\left(\frac{e^{-c\xi} e^{-c\eta}}{n^{\tau}}\right)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{1.17}$$

where

$$2\big(K^{(4)}\big)_{11}(\xi, \eta) = 2\big(K^{(4)}\big)_{22}(\eta, \xi) = K_{\mathrm{Ai}}(\xi, \eta) - \frac{1}{2}\mathrm{Ai}(\xi)\int_{\eta}^{\infty} \mathrm{Ai}(s) ds,$$

$$2\big(K^{(4)}\big)_{12}(\xi, \eta) = -\frac{\partial}{\partial \eta} K_{\mathrm{Ai}}(\xi, \eta) - \frac{1}{2}\mathrm{Ai}(\xi)\mathrm{Ai}(\eta),$$

$$2\big(K^{(4)}\big)_{21}(\xi, \eta) = -\int_{\xi}^{\infty} K_{\mathrm{Ai}}(s, \eta)\, ds + \frac{1}{2}\int_{\xi}^{\infty} \mathrm{Ai}(s)\, ds \int_{\eta}^{\infty} \mathrm{Ai}(s)\, ds.$$

(iii) *The case $\beta = 1$:*

$$\frac{1}{\lambda_n^2} K_{n,1}^{(\lambda_n)}\big(\xi^{(n)}, \eta^{(n)}\big) = K^{(1)}(\xi, \eta) + \mathcal{O}(n^{-\tau})\begin{pmatrix} e^{-c\xi} & e^{-c\xi} e^{-c\eta} \\ e^{-c\min(\xi,\eta)} & e^{-c\eta} \end{pmatrix}, \tag{1.18}$$

where

$$\left(K^{(1)}\right)_{11}(\xi,\eta) = \left(K^{(1)}\right)_{22}(\eta,\xi) = K_{\mathrm{Ai}}(\xi,\eta) + \frac{1}{2}\mathrm{Ai}(\xi)\int_{-\infty}^{\eta}\mathrm{Ai}(s)\,ds,$$

$$\left(K^{(1)}\right)_{12}(\xi,\eta) = -\frac{\partial}{\partial\eta}K_{\mathrm{Ai}}(\xi,\eta) - \frac{1}{2}\mathrm{Ai}(\xi)\mathrm{Ai}(\eta),$$

$$\left(K^{(1)}\right)_{21}(\xi,\eta) = -\int_{\xi}^{\infty}K_{\mathrm{Ai}}(s,\eta)\,ds - \frac{1}{2}\int_{\xi}^{\eta}\mathrm{Ai}(s)\,ds + \frac{1}{2}\int_{\xi}^{\infty}\mathrm{Ai}(s)\,ds \times \int_{\eta}^{\infty}\mathrm{Ai}(s)ds - \frac{1}{2}\mathrm{sgn}(\xi-\eta).$$

As above we now state the consequences of this result for the l-point correlation functions and for the distribution of the largest eigenvalue.

Corollary 1.5. *With the notation of Theorem* (1.4) *and* (1.5) *and* $\lambda_n(M)$ *denoting the largest eigenvalue of* M *we have for* $l \in \mathbb{N}$, $\xi, \xi_i \in \mathbb{R}$ *that the following limits*

(a) $\lim_{n\to\infty} \frac{1}{\lambda_n^{2l}} R_{n,\beta,l}\left(\beta_n + \frac{\xi_1}{\lambda_n^2}, \ldots, \beta_n + \frac{\xi_l}{\lambda_n^2}\right)$ *for* $\beta = 1, 2$*;*

$\lim_{n\to\infty} \frac{1}{\lambda_n^{2l}} R_{n/2,4,l}\left(\beta_n + \frac{\xi_1}{\lambda_n^2}, \ldots, \beta_n + \frac{\xi_l}{\lambda_n^2}\right)$,

(b) $\lim_{n\to\infty} \mathbb{P}_{n,\beta}\left(\left\{M : \lambda_n(M) \le \beta_n + \frac{\xi}{\lambda_n^2}\right\}\right)$ *for* $\beta = 1, 2$*;*

$\lim_{n\to\infty} \mathbb{P}_{\frac{n}{2},4}\left(\left\{M : \lambda_n(M) \le \beta_n + \frac{\xi}{\lambda_n^2}\right\}\right)$,

exist (with n *even for* $\beta = 1, 4$*) and are independent of* Q *(cf.* (1.1)*).*

This Corollary can be shown to be true in exactly the same way as Corollaries 1.2 and 1.3 were proven in Ref. 8 and we will not repeat the arguments here. Comparing the statements of Theorem 1.1 in Ref. 8 with Theorem 1.4 above shows that the limits in Corollary 1.5 are exactly the same as the ones stated in Corollaries 1.2 and 1.3 of Ref. 8. This implies in particular that the limits in statement (b) are given by the celebrated Tracy–Widom distributions. (Observe also that in Ref. 8 the results were stated for cluster functions rather than for correlation functions.)

We finally turn to the spectral statistics in the bulk.

Theorem 1.6. (bulk) *(cf. Ref. 7, Theorem 1.1). Let $\beta = 1,2$ or 4, $x \in (0,1)$ and define*

$$q_n = \left(\frac{\beta_n}{n\omega_n(x)}\right)^{-1/2}, \quad q_{n,2} = q_{n,1} = q_n, \quad q_{n,4}^2 = \frac{1}{2}q_n^2, \tag{1.19}$$

Then, for $n \to \infty$ (n even for $\beta = 1,4$) the following holds uniformly for ξ, η in compact subsets of $\mathbb{R}$ and x in compact subsets of $(0,1)$.

(i) *The case $\beta = 2$:*

$$\frac{1}{q_{n,2}^2} K_n\left(\beta_n x + \frac{\xi}{q_{n,2}^2}, \beta_n x + \frac{\eta}{q_{n,2}^2}\right) = K_\infty(\xi - \eta) + \mathcal{O}\left(\frac{1}{n}\right), \tag{1.20}$$

where

$$K_\infty(t) = \frac{\sin \pi t}{\pi t}. \tag{1.21}$$

(ii) *The cases $\beta = 1$ and 4:*

$$\frac{1}{q_{n,1}^2} K_{n,1}^{(q_{n,1})}\left(\beta_n x + \frac{\xi}{q_{n,1}^2}, \beta_n x + \frac{\eta}{q_{n,1}^2}\right) = K_{\infty,1}(\xi,\eta) + \begin{pmatrix} \mathcal{O}(n^{-1/2}) & \mathcal{O}(n^{-1}) \\ \mathcal{O}(n^{-1}) & \mathcal{O}(n^{-1/2}) \end{pmatrix}, \tag{1.22}$$

$$\frac{1}{q_{n,4}^2} K_{\frac{n}{2},4}^{(q_{n,4})}\left(\beta_n x + \frac{\xi}{q_{n,4}^2}, \beta_n x + \frac{\eta}{q_{n,4}^2}\right) = K_{\infty,4}(\xi,\eta) + \begin{pmatrix} \mathcal{O}(n^{-1/2}) & \mathcal{O}(n^{-1}) \\ \mathcal{O}(n^{-1}) & \mathcal{O}(n^{-1/2}) \end{pmatrix}, \tag{1.23}$$

where

$$K_{\infty,1}(\xi,\eta) = \begin{pmatrix} K_\infty(\xi - \eta) & \frac{\partial}{\partial \xi} K_\infty(\xi - \eta) \\ \int_0^{\xi - \eta} K_\infty(s)ds - \frac{1}{2}\operatorname{sgn}(\xi - \eta) & K_\infty(\eta - \xi) \end{pmatrix}, \tag{1.24}$$

$$K_{\infty,4}(\xi,\eta) = \begin{pmatrix} K_\infty(2(\xi - \eta)) & \frac{\partial}{\partial \xi} K_\infty(2(\xi - \eta)) \\ \int_0^{\xi - \eta} K_\infty(2s)ds & K_\infty(2(\eta - \xi)) \end{pmatrix}. \tag{1.25}$$

Again we state the consequences of this theorem for the l-point correlation functions and for gap probabilities.

Corollary 1.7. *With the notation of Theorem* 1.6 *and* (1.15) *we have for* $l \in \mathbb{N}$, $x \in (0, 1)$, $\xi, \xi_i \in \mathbb{R}$ *that the following limits*

$$\text{(a)} \lim_{n\to\infty} \frac{1}{q_{n,\beta}^{2l}} R_{n,\beta,l}\left(\beta_n x + \frac{\xi_1}{q_{n,\beta}^2}, \ldots, \beta_n x + \frac{\xi_l}{q_{n,\beta}^2}\right) \quad \textit{for } \beta = 1, 2;$$

$$\lim_{n\to\infty} \frac{1}{q_{n,4}^{2l}} R_{n/2,4,l}\left(\beta_n x + \frac{\xi_1}{q_{n,4}^2}, \ldots, \beta_n x + \frac{\xi_l}{q_{n,4}^2}\right),$$

$$\text{(b)} \lim_{n\to\infty} \mathbb{P}_{n,\beta}\left(\left\{M\colon \textit{ no eigenvalue of } M \textit{ lies in } \left(\beta_n x - \frac{\xi}{q_{n,\beta}^2}, \beta_n x + \frac{\xi}{q_{n,\beta}^2}\right)\right\}\right) \quad \textit{for } \beta = 1, 2;$$

$$\lim_{n\to\infty} \mathbb{P}_{\frac{n}{2},4}\left(\left\{M\colon \textit{ no eigenvalue of } M \textit{ lies in } (\beta_n x - \frac{\xi}{q_{n,4}^2}, \beta_n x + \frac{\xi}{q_{n,4}^2})\right\}\right)$$

exist (with n even for $\beta = 1, 4$*) and are independent of Q (cf.* (1.1)*).*

For a proof and a description of the limits, see the corresponding results, Corollaries 1.2 and 1.3, in Ref. 7. We would like to stress again that the limiting local spectral statistics of Hermite-type ensembles as considered in Refs. 7, 8 agree in the bulk and at the soft spectral edge exactly with those for Laguerre-type ensembles considered in the present paper.

We conclude the Introduction with a brief outline of the remaining parts of this paper. In Sec. 2 we derive formulae (see Theorem 2.7, Lemma 2.10, Corollary 2.15) for the scalar functions $S_{n,\beta}$, $\beta = 1, 4$, appearing in the definition of the matrix kernels $K_{n,\beta}$ in (1.9), (1.10), in terms of orthogonal polynomials. Here we follow mostly.[(7,8,25)] The precise form of the relation (2.40) in Proposition 2.9 below and the skew symmetry of G_{11} and $\hat{G}_{11}$ reported in Lemma 2.10(ii), are extremely useful in proving precise error estimates at various points in this paper. Relation (2.40) and the skew symmetry in Lemma 2.10(ii), can also be used to improve some of the error estimates in Refs. 7, 8 (cf. Remark 4.1 in Ref. 8). At the end of Sec. 2 we have all the necessary ingredients to formulate the strategy for proving our main results (see Remark 2.16).

As in Refs. 7, 8 one crucial step in the analysis is to show the invertibility of a certain $m \times m$ matrix (see T_m in (2.49) below), where m denotes the degree of the polynomial Q. This will be done in Sec. 3. Here estimates (essentially) derived in Ref. 5, 7 are very useful (see Propositions 3.4, 3.5, 3.6). However, the proof of the invertibility of the $m \times m$ matrix T_m in the present situation, is considerably more complicated than the analogous situation in Ref. 7, 8, and new ingredients, over and above the estimates in Ref. 5, 7, are needed.

Sections 4 and 5 provide all the asymptotic information on the orthogonal polynomials needed in this paper. We start the analysis from the pointwise asymptotic results derived in Ref. 23 by a Riemann–Hilbert (RH) steepest-descent analysis. In Sec. 4 we reformulate these asymptotic results in such a way that they can be conveniently used in the subsequent sections. Note that our splitting of $\mathbb{R}_+$ into intervals with different leading asymptotics, differs from the one used in Ref. 7, and leads to improved error estimates, in particular see Lemma 2.6 below. In Sec. 5 we then derive asymptotic formulae for integrals of the functions ϕ_k defined in (1.7) and of various related functions. Most of these calculations are needed to determine the leading order behavior of the matrix B which appears in Widom's formalism discussed in Sec. 2.

Our final Sec. 6 combines all auxiliary results and provides proofs for our main results. Here we give all details for the hard edge case which was not present in Refs. 7, 8. For the soft edge and the bulk we do not repeat those arguments which can already be found in Refs. 7, 8.

Remark. Throughout this paper, D denotes differentiation and ε denotes the integral operator with kernel $\varepsilon(x, y) = \frac{1}{2}\text{sgn}(x - y)$. Furthermore, by $\varepsilon f(x)$ we always mean the following,

$$\varepsilon f(x) = \frac{1}{2} \int_0^\infty \text{sgn}(x - y) f(y)\, dy, \quad x > 0.$$

The property $D\varepsilon f(x) = f(x)$ is clearly true for all continuous and integrable functions f on $\mathbb{R}_+$. However, the relation $\varepsilon D f(x) = f(x)$ is only true if $f(0) = 0$. In what follows, the relevant function f will always have this property, and we will use the relation $\varepsilon D f(x) = f(x)$ without further comment.

2. WIDOM'S FORMALISM

Following[7,8,25] we will derive in this section formulae for the scalar functions $S_{n,\beta}$, $\beta = 1, 4$ appearing in the definition of the matrix kernels $K_{n,\beta}$ in (1.9), (1.10). Furthermore, we will present all properties of the terms appearing in the formulae needed to prove our main theorems, except for the asymptotic results on the orthogonal polynomials. Those results will be provided in Sec. 6.

Recall first (see Ref. 22) the following representations for $S_{n,\beta}$ corresponding to probability density functions of the form (1.2), (1.3). Let $\{r_k(x)\}_{k\geq 0}$ be any sequence of polynomials with r_k having exact degree k. For $k = 0, 1, 2, \ldots$, set

$$\psi_{k,\beta}(x) = \begin{cases} r_k(x) w_1(x), & \beta = 1 \\ r_k(x)(w_4(x))^{1/2}, & \beta = 4. \end{cases} \tag{2.1}$$

Let $M_{n,1}$ denote the $n \times n$ matrix with entries

$$(M_{n,1})_{jk} = \langle \psi_{j,1}, \varepsilon\psi_{k,1} \rangle, \quad 0 \le j, k \le n-1, \tag{2.2}$$

where we recall that ε denotes the integral operator with kernel $\varepsilon(x, y) = \frac{1}{2}\mathrm{sgn}(x - y)$ and $\langle f, h \rangle = \int_0^\infty f(x)h(x)\,dx$ is the standard real inner product on $\mathbb{R}_+$. Furthermore, denote by $M_{n,4}$ the $2n \times 2n$ matrix with entries

$$(M_{n,4})_{jk} = \langle \psi_{j,4}, \psi'_{k,4} \rangle, \quad 0 \le j, k \le 2n-1, \tag{2.3}$$

The matrices $M_{n,1}$ and $M_{n,4}$ are skew symmetric and invertible (see e.g. Ref. 2, (4.17), (4.20)). Let $\mu_{n,1}$, $\mu_{n,4}$ denote the inverses of $M_{n,1}$, $M_{n,4}$ respectively. With this notation we have the following formulae (see Ref. 22) for $S_{n,\beta}$

$$S_{n,1}(x, y) = -\sum_{j,k=0}^{n-1} \psi_{j,1}(x)\,(\mu_{n,1})_{jk}\,(\varepsilon\psi_{k,1})(y), \quad n \text{ even}, \tag{2.4}$$

$$S_{n,4}(x, y) = \sum_{j,k=0}^{2n-1} \psi'_{j,4}(x)\,(\mu_{n,4})_{jk}\,\psi_{k,4}(y). \tag{2.5}$$

As noted in (Ref. 8, (1.49), (1.50)) the following representations of $\varepsilon S_{n,\beta}$ that are convenient for the study of the (2,1)-entries of $K_{n,\beta}$ are immediate from (2.4) and (2.5).

Proposition 2.1.

$$(\varepsilon S_{n,1})(x, y) = -\int_x^y S_{n,1}(t, y)\,dt, \quad n \text{ even}, \tag{2.6}$$

$$(\varepsilon S_{n,4})(x, y) = -\int_x^y S_{n,4}(t, y)\,dt = -\int_x^\infty S_{n,4}(t, y)\,dt = \int_0^x S_{n,4}(t, y)\,dt \tag{2.7}$$

Proof: The first equation follows from (2.4) and the skew symmetry of $\mu_{n,1}$ which implies in turn the skew symmetry of $\varepsilon S_{n,1}$: In particular $\varepsilon S_{n,1}(y, y) = 0$ for all $y > 0$. The first relation of (2.7) follows from (2.5) in a similar way, using the skew symmetry of $\mu_{n,4}$ and $\varepsilon\psi'_{j,4} = \psi_{j,4}$. The remaining two equalities are consequences of $(\varepsilon S_{n,4})(+\infty, y) = 0$ for all $y > 0$ together with the trivial relations $\varepsilon f(x) = \int_0^x f(t)\,dt - \varepsilon f(+\infty) = \varepsilon f(+\infty) - \int_x^\infty f(t)\,dt$, which hold for integrable functions f. □

An essential feature of formulae (2.4), (2.5) is that the polynomials $\{r_k\}$ are arbitrary and we are free to choose them conveniently to facilitate the asymptotic analysis of (1.9), (1.10) as $n \to \infty$ (see discussion in Ref. 7, below (1.18)). Widom

[25] found that the choice of orthogonal polynomials for $\{r_k\}$ leads to particularly convenient expressions for $S_{n,\beta}$ in cases where w'_β/w_β is a rational function. In Refs. 7, 8 it was then shown how these formulae together with detailed asymptotic information on the orthogonal polynomials lead to universality results.

In order to be able to use the same set of orthogonal polynomials for $\beta = 1$, 4 (and 2) we have defined $w = w_1^2 = w_4 (= w_2)$ in (1.4), (1.5). The role of r_k, $\psi_{k,\beta}$ above is then played by p_k and ϕ_k defined in (1.7) above. The simultaneous treatment of $\beta = 1$ and 4 is further facilitated by assuming n to be even and by considering $S_{n,1}$ together with $S_{\frac{n}{2},4}$.

Consequently, let n be an *even* integer where we assume in addition that $n \geq m$ (recall from (1.6) that m denotes the degree of the polynomial $V(x) = \sum_{j=0}^m q_j x^j$). Following Widom[25] we denote

$$\mathcal{H} := \operatorname{span}(\phi_0, \phi_1, \ldots, \phi_{n-1}). \tag{2.8}$$

Following (Ref. 25, (3.3) and (3.4)) we introduce the $2m$-dimensional space

$$\mathfrak{g} := \operatorname{span}(\{x^j \phi_n(x), x^j \phi_{n-1}(x) \mid -1 \leq j \leq m-2\}).$$

From the standard three-term recurrence relation satisfied by the orthonormal functions ϕ_j (see Ref. 20), it follows directly that

$$\mathfrak{g} = \operatorname{span}\left(\{\phi_k \mid n - m + 1 \leq k \leq n + m - 2\} \cup \left\{\frac{\phi_n(x)}{x}, \frac{\phi_{n-1}(x)}{x}\right\}\right).$$

Define

$$\mathfrak{g}^{(1)} := \mathfrak{g} \cap \mathcal{H}, \quad \text{and} \quad \mathfrak{g}^{(2)} := \{f \in \mathfrak{g} \mid \langle f, h\rangle = 0, \text{ for all } h \in \mathcal{H}\}.$$

Our first task is to construct a basis for $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$. Define

$$\tilde{\psi}_1(x) := \frac{\gamma_{n-1}}{\gamma_n}\left[p_{n-1}(0)\frac{\phi_n(x)}{x} - p_n(0)\frac{\phi_{n-1}(x)}{x}\right], \tag{2.9}$$

$$\tilde{\psi}_2(x) := 2\pi i \frac{\gamma_{n-1}}{\gamma_n}\left[C(p_{n-1}w)(0)\frac{\phi_n(x)}{x} - C(p_n w)(0)\frac{\phi_{n-1}(x)}{x}\right], \tag{2.10}$$

where C denotes the Cauchy transformation, i.e.

$$C(p_j w)(0) = \frac{1}{2\pi i}\int_0^\infty \frac{p_j(y)w(y)}{y}dy.$$

Let β_n be the Mhaskar–Rakhmanov–Saff number as defined in Sec. 4.1 below, let d_n be some negative number specified in (4.24) below, and define

$$\psi_1 := \alpha d_n \sqrt{\frac{\beta_n}{n}}\tilde{\psi}_1, \quad \text{and} \quad \psi_2 := \frac{1}{d_n}\sqrt{\frac{\beta_n}{n}}\tilde{\psi}_2. \tag{2.11}$$

Furthermore, let $\Phi := (\Phi_1, \Phi_2)$ with

$$\Phi_1 := (\phi_{n-1}, \phi_{n-2}, \ldots, \phi_{n-m+1}, \tilde{\psi}_1), \quad \Phi_2 := (\phi_n, \phi_{n+1}, \ldots, \phi_{n+m-2}, \tilde{\psi}_2).$$

With this notation we can prove the following Lemma.

Lemma 2.2. *Φ_j is a basis of $\mathfrak{g}^{(j)}$ for $j = 1, 2$.*

Proof: Our approach to proving the Lemma is as follows. Assume that the following four statements are true:

(i) span $\Phi_1 \subseteq \mathfrak{g}^{(1)}$
(ii) span $\Phi_2 \subseteq \mathfrak{g}^{(2)}$
(iii) the m functions in Φ_1 are linearly independent
(iv) the m functions in Φ_2 are linearly independent.

Then it only remains to be seen that $\dim(\text{span}\,\Phi_1) = \dim \mathfrak{g}^{(1)}$ and $\dim(\text{span}\,\Phi_2) = \dim \mathfrak{g}^{(2)}$. Since $\mathfrak{g}^{(1)} \cap \mathfrak{g}^{(2)} = \{0\}$, this follows from

$$2m = \dim \mathfrak{g} \geq \dim \mathfrak{g}^{(1)} + \dim \mathfrak{g}^{(2)} \geq \dim(\text{span}\,\Phi_1) + \dim(\text{span}\,\Phi_2) = 2m.$$

We now turn to verifying the four statements (i)–(iv).

(i) One only needs to show that $\tilde{\psi}_1 \in \mathfrak{g}^{(1)}$. Applying the Christoffel–Darboux formula (see Ref. 20) to Eq. (2.9) we have

$$\tilde{\psi}_1(x) = \sum_{k=0}^{n-1} p_k(0)\phi_k(x). \tag{2.12}$$

This shows that $\tilde{\psi}_1$ is in $\mathcal{H}$ and hence in $\mathfrak{g}^{(1)}$.

(ii) We need to prove that $\int_0^\infty \phi_k(x)\tilde{\psi}_2(x)\,dx = 0$ for all $0 \leq k \leq n-1$. Write

$$\frac{\phi_k(x)}{x} = \left(q_{k-1}(x) + \frac{p_k(0)}{x}\right)\sqrt{w(x)}$$

for some polynomial q_{k-1} of degree $k-1$ (resp. $q_{-1} \equiv 0$ for $k = 0$). From orthogonality we obtain for $0 \leq k \leq n-1$,

$$\begin{aligned}\int_0^\infty \phi_k(x)\frac{\phi_{n-1}(x)}{x}\,dx &= \int_0^\infty \left(q_{k-1}(x) + \frac{p_k(0)}{x}\right) p_{n-1}(x)w(x)\,dx \\ &= p_k(0)\int_0^\infty \frac{p_{n-1}(x)w(x)}{x}\,dx \\ &= 2\pi i p_k(0)C(p_{n-1}w)(0),\end{aligned}$$

and similarly

$$\int_0^\infty \phi_k(x)\frac{\phi_n(x)}{x}dx = 2\pi i p_k(0)C(p_n w)(0).$$

This implies that for $0 \leq k \leq n-1$,

$$\int_0^\infty \phi_k(x)\tilde{\psi}_2(x)\,dx = (2\pi i)^2\frac{\gamma_{n-1}}{\gamma_n}p_k(0)[C(p_{n-1}w)(0)C(p_n w)(0) - C(p_n w)(0)C(p_{n-1}w)(0)] = 0.$$

(iii) It suffices to prove that $\tilde{\psi}_1 \notin \text{span}(\phi_{n-1}, \phi_{n-2}, \ldots, \phi_{n-m+1})$. This follows again from Eq. (2.12) as $p_0(0) \neq 0$ and $n - m + 1 > 0$.

(iv) We prove by contradiction that $\tilde{\psi}_2 \notin$ span $(\phi_n, \ldots, \phi_{n+m-2})$. Assume otherwise. Then $\lim_{x\to 0}\frac{x}{\sqrt{w(x)}}\tilde{\psi}_2(x) = 0$. On the other hand, using the Christoffel–Darboux formula and the orthogonality relations for p_k we have

$$\begin{aligned}
&\lim_{x\to 0}\frac{x}{\sqrt{w(x)}}\tilde{\psi}_2(x)\\
&= \frac{\gamma_{n-1}}{\gamma_n}\int_0^\infty\left(\frac{p_{n-1}(y)w(y)p_n(0)}{y} - \frac{p_n(y)w(y)p_{n-1}(0)}{y}\right)dy\\
&= -\sum_{k=0}^{n-1} p_k(0)\int_0^\infty p_k(y)w(y)dy\\
&= -p_0(0)^2\int_0^\infty w(y)dy = -1.
\end{aligned}$$

This proves the Lemma. □

Next we consider the operator $[D, K] = DK - KD$ which plays a central role in Ref. 25. Recall that D denotes differentiation and K denotes the orthogonal projection onto $\mathcal{H}$, i.e.

$$(Kf)(x) = \int K(x, y)f(y)\,dy, \quad \text{with } K(x, y) = \sum_{k=0}^{n-1}\phi_k(x)\phi_k(y).$$

It follows from Ref. 25 that the kernel of the operator $[D, K]$ can be expressed in terms of functions in $\mathfrak{g}$ (in fact this motivates the definition of $\mathfrak{g}$). More precisely, it is shown in Ref. 25 that there exists a $2m \times 2m$ real matrix A such that

$$[D, K]f = \Phi A\langle f, \Phi^t\rangle, \qquad \text{for all } f \in C^1(\mathbb{R}_+) \text{ with } f' \in L^1(\mathbb{R}_+). \quad (2.13)$$

Moreover A has the form

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}, \quad \text{where } A_{12} = A_{21}^t \text{ is of size } m \times m. \tag{2.14}$$

Here $\langle f, \Phi^t \rangle$ denotes the (column) vector $\int_0^\infty f(x)\Phi^t(x)\,dx$. In order to determine the entries of A we first prove the following Proposition.

Proposition 2.3. *For all integers ℓ with $0 \le \ell \le n-1$ we have*

$$[D, K]\phi_\ell = \sum_{k=n}^{n+m-2} \left(-\frac{1}{2}\langle V'\phi_\ell, \phi_k\rangle\right)\phi_k + \left(-\frac{n}{2\beta_n}\right)\langle \phi_\ell, \psi_1\rangle \psi_2.$$

Proof: Let $0 \le \ell \le n-1$. Then, since $K\phi_\ell = \phi_\ell$, we obtain

$$\begin{aligned}[D, K]\phi_\ell &= D\phi_\ell - KD\phi_\ell = (I-K)\phi_\ell' \\ &= (I-K)(p_\ell'\sqrt{w}) + \frac{\alpha}{2}(I-K)\left(\frac{\phi_\ell}{x}\right) - \frac{1}{2}(I-K)(V'\phi_\ell). \end{aligned} \tag{2.15}$$

Let $\tilde{w}(x) = \frac{1}{x}\sqrt{w(x)}$. Observe that

$$p_\ell'\sqrt{w} \in \mathcal{H}, \quad \frac{\phi_\ell}{x} \in p_\ell(0)\tilde{w} + \mathcal{H}, \quad \text{and} \quad V'\phi_\ell \in \sum_{k=n}^{n+m-2} \langle V'\phi_\ell, \phi_k\rangle\phi_k + \mathcal{H}.$$

Here the last formula follows from the fact $V'\phi_\ell \in \operatorname{span}(\phi_0, \phi_1, \ldots, \phi_{n+m-2})$. Since $(I-K)f = 0$ for $f \in \mathcal{H}$, and since $(I-K)\phi_k = \phi_k$ for $k \ge n$, we then obtain from (2.15)

$$[D, K]\phi_\ell = \frac{\alpha}{2}p_\ell(0)(I-K)(\tilde{w}) + \sum_{k=n}^{n+m-2}\left(-\frac{1}{2}\langle V'\phi_\ell, \phi_k\rangle\right)\phi_k. \tag{2.16}$$

It now remains to determine $(I-K)(\tilde{w})$. Note that

$$\begin{aligned}\tilde{\psi}_2(x) &= \frac{\gamma_{n-1}}{\gamma_n}\int_0^\infty \frac{\sqrt{w(y)}}{xy}(\phi_{n-1}(y)\phi_n(x) - \phi_n(y)\phi_{n-1}(x))\,dy \\ &= \int_0^\infty \frac{\sqrt{w(y)}}{xy}(x-y)\sum_{k=0}^{n-1}\phi_k(x)\phi_k(y)\,dy \\ &= \int_0^\infty K(x,y)\tilde{w}(y)\,dy - \frac{1}{x}\int_0^\infty K(x,y)\sqrt{w(y)}\,dy = K(\tilde{w}) - \frac{1}{x}K(\sqrt{w}).\end{aligned}$$

Since $\sqrt{w} \in \mathcal{H}$, we have $K(\sqrt{w}) = \sqrt{w}$. We then obtain $\tilde{\psi}_2 = (K - I)(\tilde{w})$, so that by (2.11),

$$(I - K)(\tilde{w}) = -\tilde{\psi}_2 = -d_n\sqrt{\frac{n}{\beta_n}}\psi_2.$$

Inserting this relation into (2.16) we obtain

$$[D, K]\phi_\ell = -\frac{1}{2}\alpha d_n p_\ell(0)\sqrt{\frac{n}{\beta_n}}\psi_2 + \sum_{k=n}^{n+m-2}\left(-\frac{1}{2}\langle V'\phi_\ell, \phi_k\rangle\right)\phi_k. \quad (2.17)$$

Finally, observe that by (2.11) and (2.12)

$$\langle \phi_\ell, \psi_1\rangle = \alpha d_n\sqrt{\frac{\beta_n}{n}}\left\langle \phi_\ell, \sum_{k=0}^{n-1} p_k(0)\phi_k\right\rangle = \alpha d_n p_\ell(0)\sqrt{\frac{\beta_n}{n}}.$$

The Proposition follows by inserting this relation into (2.17). □

Proposition 2.3 implies that for all $f \in \mathcal{H}$,

$$[D, K]f = -\sum_{k=n}^{n+m-2}\sum_{\ell=0}^{n-1}\phi_k\frac{1}{2}\langle V'\phi_\ell, \phi_k\rangle\langle f, \phi_\ell\rangle - \psi_2\frac{n}{2\beta_n}\langle f, \psi_1\rangle.$$

Note that $V'\phi_\ell \in \mathcal{H}$ for $\ell \le n - m$: Hence $\langle V'\phi_\ell, \phi_k\rangle = 0$ for $\ell \le n - m$ and $k \ge n$. Therefore,

$$\begin{aligned}[D, K]f &= -\sum_{k=n}^{n+m-2}\sum_{\ell=n-m+1}^{n-1}\phi_k\frac{1}{2}\langle V'\phi_\ell, \phi_k\rangle\langle f, \phi_\ell\rangle - \psi_2\frac{n}{2\beta_n}\langle f, \psi_1\rangle \\ &= \Phi_2\left[-\frac{n}{\beta_n}\begin{pmatrix} Q_n & 0 \\ 0 & \frac{1}{2}\end{pmatrix}\right]\langle f, \Phi_1^t\rangle, \quad \text{for } f \in \mathcal{H}, \end{aligned} \quad (2.18)$$

where Q_n is the $(m-1)\times(m-1)$ matrix given by

$$Q_n(i, j) = \frac{\beta_n}{2n}\langle V'\phi_{n-j}, \phi_{n+i-1}\rangle, \quad \text{for } 1 \le i, j \le m - 1.$$

On the other hand (2.13) and (2.14) imply

$$[D, K]f = \Phi_2 A_{21}\langle f, \Phi_1^t\rangle, \quad \text{for } f \in \mathcal{H}.$$

It is easy to see that the map $\mathfrak{g}^{(1)} \ni f \mapsto \langle f, \Phi_1^t\rangle \in \mathbb{R}^m$ is a bijection. Since $\mathfrak{g}^{(1)} \subseteq \mathcal{H}$ this shows that $\mathcal{H} \ni f \mapsto \langle f, \Phi_1^t\rangle \in \mathbb{R}^m$ is onto, which in turn proves that the matrix A_{21} is given by

$$A_{21} = -\frac{n}{\beta_n}\begin{pmatrix} Q_n & 0 \\ 0 & \frac{1}{2}\end{pmatrix}. \quad (2.19)$$

Remark 2.4. For $i + j > m$, $Q_n(i, j) = 0$ and for $i + j = m$, $Q_n(i, j) = \langle V'\phi_{n+i-m}, \phi_{n+i-1}\rangle$. But by the orthogonality properties of the ϕ_j's, $\langle V'\phi_{n+i-m}, \phi_{n+i-1}\rangle \neq 0$. It follows that the matrix A_{21}, and hence also A_{12}, is invertiare satble.

Lemma 2.5. (Asymptotics of the matrix A) *The asymptotic behavior of the matrix A_{21} as $n \to \infty$, is given by*

$$A_{21} = -\frac{n}{\beta_n}(Y + \mathcal{O}(n^{-1/m})), \quad \textit{where } Y := \begin{pmatrix} Q & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{2.20}$$

Here, Q is an $(m-1) \times (m-1)$-matrix which is given by

$$Q(i, j) := c_{i+j-1}, \quad \textit{for } 1 \leq i, j \leq m-1, \tag{2.21}$$

with

$$c_\ell := \frac{2^{2-2m}}{A_m}\binom{2m-2}{m-1-\ell}, \quad \textit{and} \quad A_m := \prod_{j=1}^{m} \frac{2j-1}{2j}. \tag{2.22}$$

Further, since $A_{12} = A_{21}^t$ and $Y = Y^t$, (2.20) *yields*

$$A_{12} = A_{21} + \mathcal{O}\left(\frac{n}{\beta_n} n^{-1/m}\right). \tag{2.23}$$

Proof: The proof uses the results in Ref. 23 on the asymptotics of the recurrence coefficients b_{n-1} and a_n appearing in the three-term recurrence relation

$$x\phi_n(x) = b_n\phi_{n+1}(x) + a_n\phi_n(x) + b_{n-1}\phi_{n-1}(x), \tag{2.24}$$

satisfied by the orthonormal functions ϕ_j. The asymptotic behavior of the recurrence coefficients as $n \to \infty$, is given by, cf. (Ref. 23, Theorem 2.1)

$$b_{n-1} = \frac{\beta_n}{4}\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right], \quad a_n = \frac{\beta_n}{2}\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right]. \tag{2.25}$$

Here, β_n is the Mhaskar–Rakhmanov–Saff number as defined in Sec. 4.1 below, and has the following asymptotic behavior, cf. (Ref. 23, Remark 2.2 and Proposition

3.4)

$$\beta_n = \left(\frac{2n}{mq_m A_m}\right)^{1/m} [1 + \mathcal{O}(n^{-1/m})], \quad A_m = \prod_{j=1}^{m} \frac{2j-1}{2j}, \tag{2.26}$$

with q_m the leading coefficient of the polynomial $V(x) = \sum_{k=0}^{m} q_k x^k$ (cf. (1.6)).

Note first that for the case $i + j > m$ it is clear that $Q_n(i, j) = 0$ as well as $c_{i+j-1} = 0$ by the standard definition of binomials with negative second entry. Next, consider the case $i + j \le m$. Since $\frac{\beta_k}{\beta_n} = 1 + \mathcal{O}\left(\frac{1}{n}\right)$ for $|k - n|$ bounded as $n \to \infty$ (see Proposition 5.8 below), it follows from (2.25) that

$$\frac{b_k}{b_{n-1}} = 1 + \mathcal{O}\left(\frac{1}{n}\right), \quad \text{and} \quad \frac{a_k}{b_{n-1}} = 2 + \mathcal{O}\left(\frac{1}{n}\right)$$

for $|k - n|$ bounded as $n \to \infty$. Using the three-term recurrence relation (2.24), one can then prove by induction on s that

$$x^s \phi_\ell(x) = b_{n-1}^s \sum_{r=0}^{2s} \binom{2s}{r} \left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right] \phi_{\ell-s+r}(x),$$

where the error bound $\mathcal{O}(1/n)$ does not depend on x, s, ℓ for $0 \le s \le m - 1$ and $n - m + 1 \le \ell \le n - 1$. It follows from this relation that for $i + j \le m$

$$\begin{aligned} Q_n(i, j) &= \frac{\beta_n}{2n} \langle V' \phi_{n-j}, \phi_{n+i-1} \rangle = \frac{\beta_n}{2n} \sum_{s=0}^{m-1} (s+1) q_{s+1} \langle x^s \phi_{n-j}, \phi_{n+i-1} \rangle \\ &= \frac{\beta_n}{2n} \sum_{s=i+j-1}^{m-1} (s+1) q_{s+1} b_{n-1}^s \binom{2s}{s + (i+j-1)} \left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right]. \end{aligned}$$

Using (2.25) and (2.26) we then arrive at the formula

$$\begin{aligned} Q_n(i, j) &= \frac{\beta_n}{2n} m q_m b_{n-1}^{m-1} \binom{2m-2}{m - 1 + (i+j-1)} [1 + \mathcal{O}(n^{-1/m})] \\ &= \frac{2^{2-2m}}{A_m} \binom{2m-2}{m - 1 - (i+j-1)} [1 + \mathcal{O}(n^{-1/m})] \\ &= c_{i+j-1} + \mathcal{O}(n^{-1/m}). \end{aligned}$$

This completes the proof of the Lemma. □

Following[25] we next define the real $2m \times 2m$ matrix

$$B = \langle \varepsilon \Phi^t, \Phi \rangle = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \tag{2.27}$$

Observe that B is skew symmetric so that

$$B_{11} = -B_{11}^t, \quad B_{21} = -B_{12}^t, \quad \text{and} \quad B_{22} = -B_{22}^t. \tag{2.28}$$

For the convenience of the reader we display the entries of the matrix B_{12}, which is given by $B_{12} = \langle \varepsilon \Phi_1^t, \Phi_2 \rangle$, more explicitly,

$$B_{12}(i,j) = \begin{cases} \langle \varepsilon \phi_{n-i}, \phi_{n+j-1} \rangle, & 1 \leq i, j \leq m-1, \\ \langle \varepsilon \psi_1, \phi_{n+j-1} \rangle, & i = m, 1 \leq j \leq m-1, \\ \langle \varepsilon \phi_{n-i}, \psi_2 \rangle, & 1 \leq i \leq m-1, j = m, \\ \langle \varepsilon \psi_1, \psi_2 \rangle, & i = j = m. \end{cases} \tag{2.29}$$

Lemma 2.6. (Asymptotics of the matrix B) *There exists $0 < \tau = \tau(m, \alpha) < 1$ such that:*

(i) *As (even) $n \to \infty$,*

$$B_{12} = \frac{\beta_n}{n}(X + \mathcal{O}(n^{-\tau})), \quad \text{where } X = \begin{pmatrix} R & v^t \\ v & 1 - \frac{1}{\sqrt{2m-1}} \end{pmatrix}. \tag{2.30}$$

Here, R is an $(m-1) \times (m-1)$ matrix and v is an $(m-1)$-dimensional row vector, which are given by

$$R(i,j) = \hat{I}(i+j-1), \quad v(j) = \sqrt{\frac{m}{2m-1}} I(j) - \frac{1}{2\sqrt{m}}, \quad \text{for } 1 \leq i, j \leq m-1, \tag{2.31}$$

with

$$\hat{I}(q) = \frac{2}{\pi} \int_0^1 \frac{\sin(q \arccos(2x-1))}{h(x)(1-x)} dx, \tag{2.32}$$

$$I(q) = \frac{2}{\pi} \int_0^1 \frac{\sin((q-1/2)\arccos(2x-1))}{h(x) x^{1/2}(1-x)} dx, \tag{2.33}$$

and $h(x)$ is expressed in terms of a particular hypergeometric ${}_2F_1$ function as follows:

$$h(x) = \sum_{k=0}^{m-1} 2 \frac{A_{m-1-k}}{A_m} x^k = \frac{4m}{2m-1} \, {}_2F_1(1, 1-m, 3/2-m; x). \tag{2.34}$$

Further, since $B_{21} = -B_{12}^t$ *and* $X = X^t$, (2.30) *yields*

$$B_{21} = -B_{12} + \mathcal{O}\left(\frac{\beta_n}{n} n^{-\tau}\right). \tag{2.35}$$

(ii) *As (even)* $n \to \infty$,

$$B_{11} = \mathcal{O}\left(\frac{\beta_n}{n}\right) = B_{22}, \quad B_{22} = -B_{11} + \mathcal{O}\left(\frac{\beta_n}{n} n^{-\tau}\right). \tag{2.36}$$

Proof: The Lemma is immediate from the results (5.69)–(5.71) in Sec. 5. One should note that for the entries of the form $\langle \varepsilon\phi_{n-i}, \psi_2\rangle$ we use the fact that $-I(-i+1) = I(i)$, which is true by definition. □

Finally we define the $2m \times 2m$ matrix C (see Ref. 25)

$$C := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + BA = \begin{pmatrix} I + B_{12}A_{21} & B_{11}A_{12} \\ B_{22}A_{21} & B_{21}A_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \tag{2.37}$$

with I the $m \times m$ identity matrix. We now have introduced all the ingredients needed to state Widom's result (Ref. 25, Theorem 2) concerning the kernels $S_{n,1}$ and $S_{\frac{n}{2},4}$ (cf. Ref. 7, (1.36), (1.37)).

Theorem 2.7. (Widom[(25)]) *The kernels* $S_{n,1}$ *and* $S_{\frac{n}{2},4}$ *are given (for n even) by*

$$S_{\frac{n}{2},4}(x,y) = K_n(x,y) - \Phi_2(x)A_{21}\varepsilon\Phi_1(y)^t - \Phi_2(x)A_{21}C_{11}^{-1}C_{12}\varepsilon\Phi_2(y)^t \tag{2.38}$$

$$S_{n,1}(x,y) = K_n(x,y) - (\Phi_1(x), 0)\cdot(AC(I - BAC)^{-1})^t \cdot (\varepsilon\Phi_1(y), \varepsilon\Phi_2(y))^t. \tag{2.39}$$

Remark 2.8. The invertibility of C_{11} in (2.38) and of $I - BAC$ in (2.39) is one of the assertions in Ref. 25 (see also Ref. 7, Remark 1.5).

To simplify the analysis in the present paper we need a better understanding of these kernels. We now establish the following interesting and very useful relation.

Proposition 2.9.

$$BAC = \begin{pmatrix} 0 & 0 \\ C_{21} & C_{22} \end{pmatrix}. \tag{2.40}$$

Proof: Using $A = A^t$, and the fact that $\varepsilon f \in C^1(\mathbb{R}_+)$, $(\varepsilon f)' = f \in L_1(\mathbb{R}_+)$ for all $f \in \mathfrak{g}$, we conclude that

$$DK\varepsilon f = KD\varepsilon f + [D,K]\varepsilon f = Kf + \Phi A\langle \varepsilon f, \Phi^t\rangle = Kf + \langle \varepsilon f, \Phi\rangle A\Phi^t,$$

for all $f \in \mathfrak{g}$. Thus,

$$DK\varepsilon\colon \mathfrak{g} \to \mathfrak{g}, \quad \text{with } (DK\varepsilon)\Phi^t = BA\Phi^t + \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \Phi^t = C\Phi^t, \tag{2.41}$$

$$DK\varepsilon - K\colon \mathfrak{g} \to \mathfrak{g}, \quad \text{with } (DK\varepsilon - K)\Phi^t = BA\Phi^t. \tag{2.42}$$

Using in addition that $\varepsilon Df = f$ for all $f \in \mathcal{H}$, we conclude

$$(BAC)\Phi^t = DK\varepsilon(DK\varepsilon - K)\Phi^t = DK\varepsilon(I - K)\Phi^t = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} C\Phi^t.$$

Since Φ is a basis of $\mathfrak{g}$ we then have

$$BAC = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} C = \begin{pmatrix} 0 & 0 \\ C_{21} & C_{22} \end{pmatrix},$$

which proves the Proposition. □

The above Proposition together with Lemma 2.10 below, restates Widom's result in a form which is particularly convenient for the asymptotic analysis in Sec. 6. Lemma 2.10 summarizes certain facts which were already used in the analysis of (Ref. 8, Sec. 4). Note, however, that some of these facts were stated in Ref. 8 in a weaker form due to the use of a different version of Proposition 2.9.

Lemma 2.8. (i) *For n even, the kernels $S_{n,1}$ and $S_{\frac{n}{2},4}$ are given by,*

$$S_{\frac{n}{2},4}(x, y) = K_n(x, y) - \Phi_2(x)A_{21}\varepsilon\Phi_1(y)^t - \Phi_2(x)G_{11}\varepsilon\Phi_2(y)^t, \tag{2.43}$$

$$S_{n,1}(x, y) = K_n(x, y) - \Phi_1(x)A_{12}\varepsilon\Phi_2(y)^t - \Phi_1(x)\hat{G}_{11}\varepsilon\Phi_1(y)^t, \tag{2.44}$$

where

$$G_{11} = A_{21}C_{11}^{-1}C_{12}, \quad \textit{and} \quad \hat{G}_{11} = -A_{12}B_{22}\hat{C}_{22}^{-t}A_{21} \quad \textit{with} \quad \hat{C}_{22} = I - C_{22}.$$

(ii) *The matricesG_{11} and $\hat{G}_{11}$ are skew symmetric. Moreover,*

$$\hat{G}_{11} = -A_{12}\hat{C}_{22}^{-1}C_{21}.$$

Proof: (i) Equation (2.43) is precisely (2.38). Next, consider the $2m \times 2m$ matrix $[AC(I - BAC)^{-1}]^t$ as a two by two block matrix with blocks of size $m \times m$ and denote the upper left and right blocks by $\hat{G}_{11}$ and $\hat{G}_{12}$, respectively. With this notation we have by (2.39),

$$S_{n,1}(x, y) = K_n(x, y) - \Phi_1(x)\hat{G}_{11}\varepsilon\Phi_1(y)^t - \Phi_1(x)\hat{G}_{12}\varepsilon\Phi_2(y)^t.$$

In order to determine $\hat{G}_{11}$ and $\hat{G}_{12}$, observe that from Proposition 2.9,

$$\begin{pmatrix} \hat{G}_{11}^t & * \\ \hat{G}_{12}^t & * \end{pmatrix} = AC \begin{pmatrix} I & 0 \\ -C_{21} & \hat{C}_{22} \end{pmatrix}^{-1} = AC \begin{pmatrix} I & 0 \\ \hat{C}_{22}^{-1} C_{21} & \hat{C}_{22}^{-1} \end{pmatrix}. \qquad (2.45)$$

Note that the invertibility of $\hat{C}_{22}$ is immediate from the invertibility of $I - BAC$. By (2.45),

$$\begin{aligned} \hat{G}_{11}^t &= (AC)_{11} + (AC)_{12} \hat{C}_{22}^{-1} C_{21} = A_{12} \big(I + C_{22} \hat{C}_{22}^{-1} \big) C_{21} \\ &= A_{12} (\hat{C}_{22} + C_{22}) \hat{C}_{22}^{-1} C_{21} = A_{12} \hat{C}_{22}^{-1} B_{22} A_{21}. \end{aligned} \qquad (2.46)$$

Since $A_{12} = A_{21}^t$ and $B_{22} = -B_{22}^t$, see (2.14) and (2.28), this yields $\hat{G}_{11} = -A_{12} B_{22} \hat{C}_{22}^{-t} A_{21}$. Further, from (2.45) we obtain,

$$\begin{aligned} \hat{G}_{12}^t &= (AC)_{21} + (AC)_{22} \hat{C}_{22}^{-1} C_{21} = A_{21} \big(C_{11} + C_{12} \hat{C}_{22}^{-1} C_{21} \big) \\ &= A_{21} + A_{21} \big(C_{11} - I + C_{12} \hat{C}_{22}^{-1} C_{21} \big). \end{aligned} \qquad (2.47)$$

From Proposition 2.9 it follows that

$$\begin{pmatrix} C_{11} - I & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ C_{21} \end{pmatrix},$$

which implies

$$C_{11} - I = -C_{12} C_{21} C_{11}^{-1}, \quad \hat{C}_{22}^{-1} C_{21} = C_{21} C_{11}^{-1}.$$

Inserting the first relation into (2.47) we obtain $\hat{G}_{12}^t = A_{21} = A_{12}^t$ and the first part of the Lemma is proven.

(ii) We will now prove that G_{11} and $\hat{G}_{11}$ are skew symmetric. Since $C_{11}^t = I - A_{12} B_{21}$, see (2.37), (2.14) and (2.28), and since $(C_{11} - I) C_{12} + C_{12} C_{22} = 0$ (which follows from $(BAC)_{12} = 0$) we have

$$B_{11} C_{11}^t A_{12} = C_{12} - C_{12} C_{22} = C_{11} C_{12} = C_{11} B_{11} A_{12}.$$

The invertibility of A_{12} (see Remark 2.4) yields $B_{11} C_{11}^{-t} = C_{11}^{-1} B_{11}$. Since $C_{12}^t = -A_{21} B_{11}$ we obtain

$$\big(A_{21} C_{11}^{-1} C_{12} \big)^t = -A_{21} B_{11} C_{11}^{-t} A_{12} = -A_{21} C_{11}^{-1} B_{11} A_{12} = -A_{21} C_{11}^{-1} C_{12}.$$

Hence $G_{11} = A_{21} C_{11}^{-1} C_{12}$ is skew symmetric.

Next, since $\hat{C}_{22}^t = I - C_{22}^t = I + A_{21}B_{12}$ and $C_{21}C_{11} + C_{22}C_{21} = C_{21}$ (which follows from $(BAC)_{21} = C_{21}$) we have,

$$B_{22}\hat{C}_{22}^t A_{21} = C_{21}C_{11} = \hat{C}_{22}C_{21} = \hat{C}_{22}B_{22}A_{21}.$$

Since $A_{21} = A_{12}^t$ is invertible we therefore have $B_{22}\hat{C}_{22}^{-t} = \hat{C}_{22}^{-1}B_{22}$ and thus

$$\hat{G}_{11} = -A_{12}\hat{C}_{22}^{-1}B_{22}A_{21} = -A_{12}\hat{C}_{22}^{-1}C_{21}. \tag{2.48}$$

The skew symmetry of $\hat{G}_{11}$ now follows from (2.46), and the Lemma is proven. □

As discussed in Remark 2.16 below, our universality results depend critically on bounds, uniform in n, for the inverse matrices C_{11}^{-1} and $\hat{C}_{22}^{-1}$ which appear in the definitions of G_{11} and $\hat{G}_{11}$ given in the previous Lemma. In order to prove the existence of such bounds we introduce

$$T_m := I - XY, \tag{2.49}$$

where the n-independent matrices X and Y were defined in Lemmas 2.6 and 2.5, respectively.

Theorem 2.11. *For all $m \geq 1$, the matrix T_m is invertible.*

Proof: For $m = 1$, the result is trivial as $X = 0$ in this case (see Lemma 2.6). For $m \geq 2$, the proof of the Theorem requires considerable detailed analysis and occupies all of Sec. 3. □

Corollary 2.12. *For all $m \geq 1$, there exists N, L such that for all $n \geq N$,*

(i) $\|C_{11}^{-1}\| = \|(I + B_{12}A_{21})^{-1}\| \leq L$
(ii) $\|\hat{C}_{22}^{-1}\| = \|(I - B_{21}A_{12})^{-1}\| \leq L.$

Proof: (i) It follows from Lemmas 2.5 and 2.6 that $I + B_{12}A_{21}$ converges to T_m as $n \to \infty$. The claim now follows from Theorem 2.11.

(ii) Since $A_{12}^t = A_{21}$ and $B_{21}^t = -B_{12}$ we have that $(I - B_{21}A_{12})^t$ converges to $I - YX$ as $n \to \infty$. Since XY and YX have the same (non-zero) eigenvalues, the invertibility of $I - YX$ follows again from Theorem 2.11, leading to statement (ii). □

Lemma 2.5, Lemma 2.6(ii) together with Corollary 2.12 imply:

Corollary 2.13. *The matrices* G_{11} *and* $\hat{G}_{11}$ *of Lemma 2.10 obey the following asymptotic bounds,*

$$G_{11} = \mathcal{O}\left(\frac{n}{\beta_n}\right), \quad \hat{G}_{11} = \mathcal{O}\left(\frac{n}{\beta_n}\right), \quad n \to \infty. \tag{2.50}$$

Note that for $m = 1$ it follows from the skew symmetry of G_{11} and $\hat{G}_{11}$, see Lemma 2.10(ii), that $G_{11} = \hat{G}_{11} = 0$.

The importance of the analog of the following observations for the proof of universality has already been noted in (Ref. 8, (1.46)).

Proposition 2.14. *With the above notation, the following statements hold true.*

(i) $A_{21}\varepsilon\Phi_1(+\infty)^t + G_{11}\varepsilon\Phi_2(+\infty)^t = 0.$

(ii) *There exists* $0 < \tau = \tau(m, \alpha) < 1$ *such that as* $n \to \infty$

$$A_{12}\varepsilon\Phi_1(+\infty)^t + \hat{G}_{11}\varepsilon\Phi_2(+\infty)^t = A_{12}\hat{C}_{22}^{-1}[\mathcal{O}(n^{-\tau})\varepsilon\Phi_1(+\infty)^t + \mathcal{O}(n^{-\tau})\varepsilon\Phi_2(+\infty)^t].$$

Proof: (i) Recall that for $f \in \mathcal{H}$ (see (2.8)) we have $0 = \frac{1}{2}\int_0^\infty f'(x)\,dx = \varepsilon(Df)(+\infty)$. Using (2.41) and $K\varepsilon\Phi^t \in \mathcal{H}^{2m}$ we obtain $0 = \varepsilon(DK\varepsilon\Phi^t)(+\infty) = \varepsilon(C\Phi^t)(+\infty) = C\varepsilon\Phi^t(+\infty)$. This implies by Lemma 2.10 that

$$\begin{aligned} &A_{21}\varepsilon\Phi_1(+\infty)^t + G_{11}\varepsilon\Phi_2(+\infty)^t \\ &\quad = A_{21}C_{11}^{-1}[C_{11}\varepsilon\Phi_1(+\infty)^t + C_{12}\varepsilon\Phi_2(+\infty)^t] = 0. \end{aligned}$$

(ii) Lemma 2.10 yields

$$A_{12}\varepsilon\Phi_1(+\infty)^t + \hat{G}_{11}\varepsilon\Phi_2(+\infty)^t = A_{12}\hat{C}_{22}^{-1}[\hat{C}_{22}\varepsilon\Phi_1(+\infty)^t - C_{21}\varepsilon\Phi_2(+\infty)^t].$$

Moreover, relations (2.23), (2.35), (2.36) together with (2.20), (2.30) imply $\hat{C}_{22} = C_{11} + \mathcal{O}(n^{-\tau})$ and $C_{21} = -C_{12} + \mathcal{O}(n^{-\tau})$ for some suitable $0 < \tau < 1$. The claim then follows from (i). □

The above Proposition together with the simple observation that for integrable functions $f \in L^1(\mathbb{R}_+)$

$$\varepsilon f(x) = \int_0^x f(s)\,ds - \varepsilon f(+\infty) = \varepsilon f(+\infty) - \int_x^\infty f(s)\,ds$$

allows us to convert (2.43) and (2.44) into a form which is particularly suitable for the analysis both at the hard and the soft edge.

Corollary 2.15. *For n even, and for some* $0 < \tau = \tau(m, \alpha) < 1$*, the kernels* $S_{n,1}$ *and* $S_{\frac{n}{2},4}$ *satisfy*

$$S_{\frac{n}{2},4}(x, y) = K_n(x, y) - \Phi_2(x) A_{21} \int_0^y \Phi_1(s)^t ds - \Phi_2(x) G_{11} \int_0^y \Phi_2(s)^t ds \tag{2.51}$$

$$= K_n(x, y) + \Phi_2(x) A_{21} \int_y^\infty \Phi_1(s)^t ds + \Phi_2(x) G_{11} \int_y^\infty \Phi_2(s)^t ds, \tag{2.52}$$

$$S_{n,1}(x, y) = K_n(x, y) - \Phi_1(x) A_{12} \left(\int_0^y \Phi_2(s)^t ds - \varepsilon \Phi_2(+\infty)^t + \varepsilon \Phi_1(+\infty)^t \right)$$
$$- \Phi_1(x) \hat{G}_{11} \left(\int_0^y \Phi_1(s)^t ds - \varepsilon \Phi_1(+\infty)^t + \varepsilon \Phi_2(+\infty)^t \right)$$
$$+ \Phi_1(x) A_{12} \hat{C}_{22}^{-1} [\mathcal{O}(n^{-\tau}) \varepsilon \Phi_1(+\infty)^t + \mathcal{O}(n^{-\tau}) \varepsilon \Phi_2(+\infty)^t] \tag{2.53}$$

$$= K_n(x, y) + \Phi_1(x) A_{12} \left(\int_y^\infty \Phi_2(s)^t ds - \varepsilon \Phi_1(+\infty)^t - \varepsilon \Phi_2(+\infty)^t \right)$$
$$+ \Phi_1(x) \hat{G}_{11} \left(\int_y^\infty \Phi_1(s)^t ds - \varepsilon \Phi_1(+\infty)^t - \varepsilon \Phi_2(+\infty)^t \right)$$
$$+ \Phi_1(x) A_{12} \hat{C}_{22}^{-1} [\mathcal{O}(n^{-\tau}) \varepsilon \Phi_1(+\infty)^t + \mathcal{O}(n^{-\tau}) \varepsilon \Phi_2(+\infty)^t]. \tag{2.54}$$

Remark 2.16. Corollary 2.15 allows us to indicate at this point which facts are essential for our proof of universality. The details of the proofs can be found in Sec. 6.

(a) *Hard edge:* For the simpler case $\beta = 4$ we see from (2.51) that $S_{\frac{n}{2},4}$ can be written as a sum of three terms. The first term is the Christoffel–Darboux kernel which we know to be universal from the analysis of the case $\beta = 2$.[23] The key to understanding the second term is the observation (cf. Propositions 6.4, 6.5) that after rescaling $\Phi_2(x)$ and $\int_0^y \Phi_1(s) ds$ are both, to leading order, scalar multiples of the vector $\mathbf{e} := (0, \dots, 0, 1)$ where the scalar factors can be expressed in terms of some Bessel functions which only depend on α. Moreover, $\mathbf{e} A_{21} \mathbf{e}^t$ just reproduces the (m, m) entry of A_{21}, which by (2.19) is (universally) given by $-\frac{n}{2\beta_n}$. By similar reasoning the leading order behavior of the last term of (2.51) is given by $\mathbf{e} G_{11} \mathbf{e}^t$ which is equal to 0 by the skew symmetry of G_{11}. The vanishing of this term by skew symmetry is fortunate since an explicit evaluation of the asymptotics of the matrix G_{11} for general m is a formidable problem. The heart of the problem is then to estimate the inverse matrix C_{11}^{-1} uniformly in n (cf. Corollary 2.12).

For $\beta = 1$ we use formula (2.53) which is a sum of four terms. The first term is the Christoffel–Darboux kernel and the last term is of lower order due to the $\mathcal{O}(n^{-\tau})$ estimate. As in the case $\beta = 4$ one can show by corresponding asymptotic formulae for the expressions depending on Φ, that the leading order behavior is given by $\mathbf{e}A_{12}\mathbf{e}^t$ and $\mathbf{e}\hat{G}_{11}\mathbf{e}^t$. The latter term vanishes by skew symmetry of $\hat{G}_{11}$ and the first term equals $-\frac{n}{2\beta_n}$ since $A_{12} = A_{21}^t$ by (2.14).

(b) *Soft edge:* The arguments here are quite similar to the ones given for the hard edge with (2.51), (2.53) replaced by (2.52) and (2.54) respectively. The most distinctive difference from the hard edge case is that the vector $\mathbf{e}$ is now replaced by $\mathbf{a} = (1, \dots, 1, \sqrt{\frac{m}{2m-1}})$. We still have the vanishing of $\mathbf{a}G_{11}\mathbf{a}^t$ and $\mathbf{a}\hat{G}_{11}\mathbf{a}^t$ by skew symmetry. However, the universality result at the soft edge hinges on the relation

$$\mathbf{a}A_{21}\mathbf{a}^t = \mathbf{a}A_{12}\mathbf{a}^t = -\frac{n}{\beta_n}\left(\frac{m}{2} + \mathcal{O}(n^{-1/m})\right)$$

of Proposition 6.7. This relation follows from the leading order evaluation

$$\mathbf{a}Y\mathbf{a}^t = \frac{m}{2}$$

which by the defintion of Y in (2.20) is based for each m on some identity for sums of binomial coefficients. It is somewhat surprising and maybe unsatisfactory that the derivation of the universal Tracy–Widom distributions at the soft edge depends on such special identities. A similar situation already appeared in (Ref. 8, (4.13) and below).

(c) *Bulk:* As in Ref. 7 the proof of universality in the bulk is less subtle than at the edges, because one can show that the Christoffel–Darboux kernel K_n dominates in (2.43) and (2.44) and the remaining two correction terms in each formula are of lower order as $n \to \infty$.

3. INVERTIBILITY OF T_m FOR $m \geq 2$

In this section we will always assume $m \geq 2$. Our objective is to prove that for such m the $m \times m$ matrices $T_m = I - XY$, defined in (2.49), are invertible. A crucial step in the proof of this result is provided by the estimates in Lemma 3.2 for the entries of the matrix X. Our proof of the basic Lemma 3.2 in Sec. 3.2 follows closely the corresponding proofs in Refs. 5, 7, see in particular Proposition 3.4 below.

However, as mentioned above, we face new difficulties in the Laguerre-type case which are not present for Hermite-type ensembles. In the Hermite-type case the authors show that, for any $m \geq 1$, as a map from l_∞ to l_∞, the analog of $T_m - I$ has the norm <1, and hence T_m is invertible. In the present situation, however, the last row and column in X and Y, which have no analogue in the Hermite-type case, force the matrix XY to have norm ≥ 1 for any operator norm on $\mathbb{R}^m$. Thus we may not simply invert T_m by a Neumann series and one must take a different approach. This approach is presented below and in Sec. 3.1.

We use the following representation of T_m which is immediate from (2.20), (2.30) and (2.49):

$$T_m = I - XY = \begin{pmatrix} I - RQ & -\frac{1}{2}v^t \\ -vQ & \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{2m-1}} \end{pmatrix},$$

where Q is defined in Lemma 2.5, and where R and v are defined in Lemma 2.6. The approach we follow to prove that T_m is invertible is based on the following fact. A matrix T written in block form

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if both the matrices a and $d - ca^{-1}b$ are invertible. Therefore it suffices to prove that the following two conditions (1) and (2) are satisfied.

(1) $I - RQ$ is invertible
(2) $1 + \frac{1}{\sqrt{2m-1}} - vQ(I - RQ)^{-1}v^t \neq 0$

In Sec. 3.1 we will show how these two conditions follow from the technical Lemmas 3.2 and 3.3. These Lemmas will then be proven in Sec. 3.2 and 3.3.

3.1. Proof of Conditions (1) and (2)

We introduce some convenient notation. Let R_1 and U_0 be the following $(m-1) \times (m-1)$ matrices,

$$R_1 = R - \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad U_0 = I - \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix} Q = \begin{pmatrix} \gamma & -\frac{1}{4}u \\ 0 & I \end{pmatrix}. \quad (3.1)$$

Here, $\gamma = 1 - \frac{c_1}{4}$ and $u = (c_2, \ldots, c_{m-1})$ (cf. (2.22)). Further, define $\hat{Q} = QU_0^{-1}$. It is clear that U_0 is invertible with inverse,

$$U_0^{-1} = \begin{pmatrix} \frac{1}{\gamma} & \frac{1}{4\gamma}u \\ 0 & I \end{pmatrix}.$$

Then, since $1 + \frac{c_1}{4\gamma} = \frac{1}{\gamma}$, we have

$$\hat{Q} = QU_0^{-1} = \frac{1}{\gamma}\begin{pmatrix} c_1 & u \\ u^t & \tilde{Q} \end{pmatrix}, \qquad \tilde{Q}(i,j) = \frac{c_i c_j}{4} + \gamma c_{i+j-1}, \text{ for } 2 \le i, j \le m-1. \tag{3.2}$$

With the above notation it is straightforward to check that

$$I - RQ = U_0 - R_1 Q = (I - R_1 \hat{Q})U_0. \tag{3.3}$$

Hence condition (1) is equivalent to the invertibility of $(I - R_1\hat{Q})$. Assuming condition (1) and using in addition that $\hat{Q}$ is a symmetric matrix and that $(I - R_1\hat{Q})^{-1} = I + (I - R_1\hat{Q})^{-1}R_1\hat{Q}$ we find

$$vQ(I - RQ)^{-1}v^t = (v\hat{Q})[(I - R_1\hat{Q})^{-1}R_1](v\hat{Q})^t + v\hat{Q}v^t. \tag{3.4}$$

Remark 3.1. In order to prove conditions (1) and (2) we will make use of the following norms. If A is a $p \times p$ matrix and x a row vector of size p, we define

$$\|A\|_{1\to\infty} := \max_{i,j} |A_{ij}|, \quad \|A\|_{\infty\to\infty} := \max_i \sum_k |A_{ik}|, \quad \|A\|_{\infty\to 1} := \sum_{i,j} |A_{ij}|,$$

$$\|x\|_1 := \sum_i |x_i|, \quad \|x\|_\infty := \max_i |x_i|.$$

Note that $\|\cdot\|_{1\to\infty}$ and $\|\cdot\|_{\infty\to\infty}$ are precisely the operator norms for linear maps $\ell_1(\mathbb{R}^p) \to \ell_\infty(\mathbb{R}^p)$ and $\ell_\infty(\mathbb{R}^p) \to \ell_\infty(\mathbb{R}^p)$, respectively, whereas $\|\cdot\|_{\infty\to 1}$ is merely an upper bound on the operator norm for linear maps $\ell_\infty(\mathbb{R}^p) \to \ell_1(\mathbb{R}^p)$. These observations imply the following inequalities, which are readily verified:

$$\|AB\|_{\infty\to\infty} \le \|B\|_{\infty\to 1}\|A\|_{1\to\infty}, \qquad \|AB\|_{1\to\infty} \le \|B\|_{1\to\infty}\|A\|_{\infty\to\infty},$$
$$|xAx^t| \le \|A\|_{1\to\infty}\|x\|_1^2, \qquad \|AB\|_{\infty\to\infty} \le \|A\|_{\infty\to\infty}\|B\|_{\infty\to\infty}.$$

The following two Lemmas are the key ingredients in proving that conditions (1) and (2) are satisfied.

Lemma 3.2. *The functions I and $\hat{I}$ defined by* (2.33) *and* (2.32)*, respectively, satisfy for all $m \ge 2$ and $q \ge 1$,*

(a) $|I(q) - \frac{1}{2}\delta_{1,q}| \le \frac{D}{2m}$ *with* $D = 2.22$
(b) $|\hat{I}(q) - \frac{1}{4}\delta_{1,q}| \le \frac{C}{2m}$ *with* $C = 2.18$.

Lemma 3.3. *For all $m \geq 2$,*

(a) $\|\hat{Q}\|_{\infty\to 1} \leq m(\frac{\pi}{12} + \frac{1}{2})$
(b) $\|v\hat{Q}\|_1 \leq 0.3918\sqrt{m}$
(c) $v\hat{Q}v^t < \frac{1}{\sqrt{2m-1}}$.

These Lemmas will be proven in the next two subsections.

Proof of Conditions (1) and (2): In order to prove condition (1), it follows from (3.3) that we need to show that $I - R_1\hat{Q}$ is invertible. This is done by proving that $\|R_1\hat{Q}\|_{\infty\to\infty} < 1$. From the definition of R_1 and from Lemma 3.2(b) it follows that $\|R_1\|_{1\to\infty} \leq \frac{C}{2m}$. From Remark 3.1 and Lemma 3.3(a) we then conclude

$$\|R_1\hat{Q}\|_{\infty\to\infty} \leq \|\hat{Q}\|_{\infty\to 1}\|R_1\|_{1\to\infty} \leq \frac{C}{2}\left(\frac{\pi}{12} + \frac{1}{2}\right) \leq 0.381C < 1. \quad (3.5)$$

This proves that condition (1) is satisfied. Moreover we obtain the bound

$$\|(I - R_1\hat{Q})^{-1}\|_{\infty\to\infty} \leq \frac{1}{1 - 0.381C}. \quad (3.6)$$

It remains to prove condition (2). From Eq. (3.4) and Lemma 3.3(c) it suffices to show that

$$|(v\hat{Q})[(I - R_1\hat{Q})^{-1}R_1](v\hat{Q})^t| \leq 1.$$

Using Remark 3.1, Eq. (3.6) and Lemma 3.3(b) we obtain

$$\begin{aligned} |(v\hat{Q})[(I - R_1\hat{Q})^{-1}R_1](v\hat{Q})^t| &\leq \|(I - R_1\hat{Q})^{-1}R_1\|_{1\to\infty}\|v\hat{Q}\|_1^2 \\ &\leq \|R_1\|_{1\to\infty}\|(I - R_1\hat{Q})^{-1}\|_{\infty\to\infty}\|v\hat{Q}\|_1^2 \\ &\leq \frac{C}{2}\frac{1}{1 - 0.381C}0.3918^2 < 1. \end{aligned} \quad (3.7)$$

Hence condition (2) is satisfied as well. □

Thus the invertibility of T_m follows from Lemmas 3.2 and 3.3. In the remainder of this Section we will prove that these two Lemmas are true.

3.2. Proof of Lemma 3.2

Our proof follows the corresponding parts of (Ref. 7, Sec. 6) and its improved version in Ref. 5. Define for $x \in [0, 1]$ the auxiliary function u as,

$$u(x) = \frac{1}{h(x^2)} - \frac{1 - x^2}{2} + \frac{1}{4m}, \quad (3.8)$$

where $h(x) = \frac{4m}{2m-1}\, {}_2F_1(1, -m+1; -m+3/2; x)$. Note that this function u coincides with the function u defined in (Ref. 5, (16)). We will use the following result.

Proposition 3.4. (Ref. 5, Lemma 3) *For all $m \geq 2$ the following holds.*

(a) *There exists $x_m \in (0, 1)$ such that $u' < 0$ on $[0, x_m)$ and $u' > 0$ on $(x_m, 1]$.*
(b) *$u(0) = 0$, $u(1) = \frac{1}{2m}$ and $u(x_m) > -\frac{1}{4m}$.*

3.2.1. Part (a) of Lemma 3.2

In order to analyze $I(q)$ defined by (2.33), we apply the substitution $\theta = \frac{1}{2}\arccos(2x-1)$ and use (3.8) to arrive at

$$I(q) = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} V_q(\theta)\frac{1}{h(\cos^2\theta)} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} V_q(\theta)\left(u(\cos\theta) + \frac{\sin^2\theta}{2} - \frac{1}{4m}\right)d\theta,$$

where V_q is the function,

$$V_q(\theta) \equiv \frac{\sin(2q-1)\theta}{\sin\theta} = 1 + 2\sum_{k=1}^{q-1}\cos(2k\theta). \tag{3.9}$$

Using the elementary facts,

$$\int_0^{\frac{\pi}{2}} V_q(\theta)\sin^2\theta d\theta = \frac{\pi}{4}\delta_{1,q}, \quad \text{and} \quad \int_0^{\frac{\pi}{2}} V_q(\theta)d\theta = \frac{\pi}{2},$$

integrating by parts, using the fact that $u(0) = 0$ (see Proposition 3.4) we obtain,

$$\begin{aligned} I(q) - \frac{1}{2}\delta_{1,q} &= \frac{4}{\pi}\int_0^{\pi/2} V_q(\theta)u(\cos\theta)d\theta - \frac{1}{2m}, \\ &= \int_0^{\pi/2} W_q(\theta)u'(\cos\theta)\sin\theta d\theta - \frac{1}{2m}, \quad \text{for all } q \geq 1, \end{aligned} \tag{3.10}$$

with W_q the auxiliary function,

$$W_q(\theta) = \frac{4}{\pi}\int_0^{\theta} V_q(s)ds = \frac{4}{\pi}\left(\theta + \sum_{k=1}^{q-1}\frac{\sin(2k\theta)}{k}\right), \quad \theta \in [0, \infty). \tag{3.11}$$

Here, the expression of W_q as a sum follows from (3.9). In order to prove Lemma 3.2(a) we will make use of Eq. (3.10), together with Proposition 3.4 and the following result.

Proposition 3.5. (cf. Ref. 5, Lemma 4) *Let $q \geq 1$. There exists $\theta_q \in (0, \frac{\pi}{2})$ such that the following holds.*

(a) *W_q is increasing on $[0, \theta_q]$ and $0 \leq W_q(\theta) \leq W_q(\theta_q) = 1.7$ for $\theta \in [0, \theta_q]$.*
(b) *For $\theta \in [\theta_q, \frac{\pi}{2}]$ we have $1.7 \leq W_q(\theta) \leq 2.44$.*

Proof: We distinguish three cases. First, in case $q = 1$, we have $W_1(\theta) = \frac{4}{\pi}\theta$. Then the Proposition is true with $\theta_1 = \frac{1.7\pi}{4}$. Next, consider the case $q = 2$. It follows from (3.11) that $W_2(\theta) = \frac{4}{\pi}(\theta + \sin 2\theta)$ and so W_2 is increasing on $[0, \pi/3]$ and decreasing on $[\pi/3, \pi/2]$. Since $W_2(\pi/3) = 4/3 + 2\sqrt{3}/\pi \in [1.7, 2.44]$ and $W_2(\pi/2) = 2$ we can define θ_2 to be the unique number in $[0, \pi/3]$ such that $W_2(\theta_2) = 1.7$.

Finally, we prove that the Proposition is satisfied for $q \geq 3$ as well. Define a sequence $s_k = k\frac{\pi}{2q-1}$ for integers $k \geq 0$. We first prove that

$$W_q(s_1) \leq 2.44 \quad \text{and} \quad W_q(s_2) \geq 1.7, \quad \text{for } q \geq 3. \tag{3.12}$$

Note that

$$W_q(s_1) = \frac{4}{\pi}\int_0^{\pi} \frac{\sin t}{(2q-1)\sin(\frac{t}{2q-1})}dt,$$

and that for every $t \in [0, \pi]$, $(2q-1)\sin(\frac{t}{2q-1})$ increases in q. Then $W_q(s_1)$ decreases in q, so that for all $q \geq 3$,

$$W_q(s_1) \leq W_2(\pi/3) \leq 2.44.$$

We now turn to the lower estimate on $W_q(s_2)$ for $q \geq 3$. We use $\sin(\frac{t}{2q-1}) \leq \frac{t}{2q-1}$ for $t \geq 0$ and arrive at

$$\begin{aligned} W_q(s_2) &= \frac{4}{\pi}\int_0^{\pi} \frac{\sin t}{(2q-1)\sin(\frac{t}{2q-1})}dt + \frac{4}{\pi}\int_{\pi}^{2\pi} \frac{\sin t}{(2q-1)\sin(\frac{t}{2q-1})}dt \\ &\geq \frac{4}{\pi}\int_0^{\pi} \frac{\sin t}{t} + \frac{4}{\pi}\int_{\pi}^{2\pi} \frac{\sin t}{(2q-1)\sin(\frac{t}{2q-1})}dt. \end{aligned}$$

Since the last integral is increasing in q we then have for $q \geq 3$,

$$\begin{aligned} W_q(s_2) &\geq \frac{4}{\pi}\mathrm{Si}(\pi) + \frac{4}{\pi}\int_{\pi}^{2\pi} \frac{\sin t}{5\sin(\frac{t}{5})}dt = \frac{4}{\pi}\mathrm{Si}(\pi) + \frac{4}{\pi}\int_{\pi/5}^{2\pi/5} \frac{\sin 5t}{\sin t}dt \\ &= \frac{4}{\pi}\mathrm{Si}(\pi) + W_3(2\pi/5) - W_3(\pi/5). \end{aligned}$$

The last quantity can be estimated from below using $\mathrm{Si}(\pi) \geq 1.851$ (see e.g., Ref. 1) and the explicit expression (3.11) for W_3. We then find that $W_q(s_2) \geq 1.7$ for all $q \geq 3$.

Using (3.12) we will now complete the proof of the Proposition. It is immediate that W_q is increasing on $[s_{2k}, s_{2k+1}]$ and decreasing on $[s_{2k+1}, s_{2k+2}]$. Furthermore, the monotonicity of $1/\sin\theta$ on $[0, \pi/2]$, together with (3.12) implies the following inequalities for the local maxima and minima of W_q.

$$2.44 \geq W_q(s_1) \geq W_q(s_3) \geq \cdots \geq W_q(s_{2k_1+1}), \quad \text{with } k_1 = \left[\frac{2q-3}{4}\right],$$

$$1.7 \leq W_q(s_2) \leq W_q(s_4) \leq \cdots \leq W_q(s_{2k_2}), \quad \text{with } k_2 = \left[\frac{2q-1}{4}\right].$$

Using in addition that $W_q(0) = 0$ and that $W_q(\frac{\pi}{2}) = 2$ the Proposition now follows by choosing θ_q to be the unique number in the interval $[0, s_1]$ satisfying $W_q(\theta_q) = 1.7$. Such a number exists since $W_q(0) = 0 < 1.7$ and $W_q(s_1) \geq W_q(s_2) \geq 1.7$. □

Proof of Lemma 3.2(a): From (3.10) we have

$$I(q) - \frac{1}{2}\delta_{1,q} = \int_0^{\theta^*} W_q(\theta) u'(\cos\theta) \sin\theta d\theta + \int_{\theta^*}^{\pi/2} W_q(\theta) u'(\cos\theta) \sin\theta d\theta - \frac{1}{2m}, \tag{3.13}$$

where $\theta^* \in [0, \frac{\pi}{2}]$ is defined such that $\cos\theta^* = x_m$ (see Proposition 3.4). With this choice of θ^* we have from Proposition 3.4(a) that

$$u'(\cos\theta) \begin{cases} >0, & \text{for } \theta \in [0, \theta^*], \\ <0, & \text{for } \theta \in [\theta^*, \frac{\pi}{2}]. \end{cases} \tag{3.14}$$

Since $0 \leq W_q(\theta) \leq 2.44$ for all $\theta \in [0, \frac{\pi}{2}]$, we then obtain from (3.13) and Proposition 3.4(b) that,

$$I(q) - \frac{1}{2}\delta_{1,q} \geq \int_{\theta^*}^{\pi/2} W_q(\theta) u'(\cos\theta) \sin\theta d\theta - \frac{1}{2m} \geq 2.44\, u(\cos\theta^*) - \frac{1}{2m} \geq -\frac{2.22}{2m}.$$

This is the desired lower estimate. In order to obtain the upper estimate we distinguish two cases. Consider first the case that q is such that $\theta^* \leq \theta_q$ (here θ^* is defined as above and θ_q is chosen as in Proposition 3.5). Then, since $W_q(\theta) \leq W_q(\theta^*) \leq 1.7$ for $\theta \in [0, \theta^*]$ and $W_q(\theta) \geq W_q(\theta^*)$ for $\theta \in [\theta^*, \frac{\pi}{2}]$, we obtain from (3.13), (3.14) and Proposition 3.4(b),

$$
\begin{aligned}
I(q) - \frac{1}{2}\delta_{1,q} &\leq W_q(\theta^*)\int_0^{\theta^*} u'(\cos\theta)\sin\theta d\theta \\
&\quad + W_q(\theta^*)\int_{\theta^*}^{\pi/2} u'(\cos\theta)\sin\theta d\theta - \frac{1}{2m} \\
&= -W_q(\theta^*)u(\cos\theta)\Big|_0^{\pi/2} - \frac{1}{2m} = W_q(\theta^*)\frac{1}{2m} - \frac{1}{2m} \\
&\leq \frac{0.7}{2m} \leq \frac{2.22}{2m}.
\end{aligned}
$$

Next, consider the case that q is such that $\theta^* \geq \theta_q$. Then, since $W_q(\theta) \leq 2.44$ for $\theta \in [0, \theta^*]$ and $W_q(\theta) \geq 1.7$ for $\theta \in [\theta^*, \frac{\pi}{2}]$, we obtain from (3.13), (3.14) and Proposition 3.4(b),

$$
\begin{aligned}
I(q) - \frac{1}{2}\delta_{1,q} &\leq -2.44\, u(\cos\theta)\Big|_0^{\theta^*} - 1.7\, u(\cos\theta)\Big|_{\theta^*}^{\pi/2} - \frac{1}{2m} \\
&= u(x_m)(1.7-2.44) + \frac{2.44}{2m} - \frac{1}{2m} \leq \frac{0.74}{4m} + \frac{1.44}{2m} = \frac{1.81}{2m} \leq \frac{2.22}{2m}.
\end{aligned}
$$

This proves part (a) of Lemma 3.2. □

3.2.2. Part (b) of Lemma 3.2

The proof of part (b) is analogous to the proof of part (a). In this case we introduce the function

$$
\hat{V}_q(\theta) \equiv \frac{\sin(2q\theta)\cos\theta}{\sin\theta} = \frac{1}{2}(V_{q+1}(\theta) + V_q(\theta)). \tag{3.15}
$$

It is then straightforward to check that

$$
\begin{aligned}
\hat{I}(q) - \frac{1}{4}\delta_{1,q} &= \frac{4}{\pi}\int_0^{\pi/2} \hat{V}_q(\theta)u(\cos\theta)d\theta - \frac{1}{2m} \\
&= \int_0^{\pi/2} \hat{W}_q(\theta)u'(\cos\theta)\sin\theta\, d\theta - \frac{1}{2m},
\end{aligned} \tag{3.16}
$$

where $\hat{W}_q$ is the auxiliary function,

$$
\hat{W}_q(\theta) = \frac{4}{\pi}\int_0^{\theta} \hat{V}_q(s)\, ds, \quad \theta \in [0, \infty),
$$

which satisfies the following Proposition.

Proposition 3.6. *Let $q \geq 1$. There exists $\theta_q \in (0, \frac{\pi}{2})$ such that the following holds.*

(a) *$\hat{W}_q$ is increasing on $[0, \theta_q]$ and $0 \leq \hat{W}_q(\theta) \leq \hat{W}_q(\theta_q) = 1.7$ for $\theta \in [0, \theta_q]$.*
(b) *For $\theta \in [\theta_q, \frac{\pi}{2}]$ we have $1.7 \leq \hat{W}_q(\theta) \leq 2.36$.*

Proof: The proof is similar to the proof of Proposition 3.5. Again the case $q = 1$ is trivial since $\hat{W}_1$ is monotone increasing on $[0, \frac{\pi}{2}]$ with $\hat{W}_1(0) = 0$ and $\hat{W}_1(\frac{\pi}{2}) = 2$.

In order to deal with the case $q \geq 2$ we define $t_k := k\frac{\pi}{2q}$, $k \geq 0$, where $\hat{W}_q$ attains its local extrema. Using the same arguments as in the proof of Proposition 3.5 (note that $1/\tan t$ is decreasing for $t \in (0, \frac{\pi}{2})$) it suffices to show that the following estimates hold:

(i) $\hat{W}_q(t_1) \leq 2.36$
(ii) $\hat{W}_q(t_2) \geq 1.7$.

In order to prove these two claims we use the fact that for every $t \in [0, 2\pi)$ the value of $2q \tan \frac{t}{2q}$ decreases in q (for $q \geq 2$) and converges to t as q tends to ∞. This implies that for all $q \geq 2$ we have

$$\hat{W}_q(t_1) = \frac{4}{\pi}\int_0^\pi \frac{\sin t}{2q \tan \frac{t}{2q}} dt \leq \frac{4}{\pi}\int_0^\pi \frac{\sin t}{t} dt = \frac{4}{\pi} \operatorname{Si}(\pi) \leq 2.36,$$

and

$$\begin{aligned}\hat{W}_q(t_2) &= \frac{4}{\pi}\left(\int_0^\pi \frac{\sin t}{2q \tan \frac{t}{2q}} dt + \int_\pi^{2\pi} \frac{\sin t}{2q \tan \frac{t}{2q}} dt\right) \\ &\geq \hat{W}_2(t_1) + \frac{4}{\pi}\int_\pi^{2\pi} \frac{\sin t}{t} dt = 1 + \frac{4}{\pi}(1 + \operatorname{Si}(2\pi) - \operatorname{Si}(\pi)) \geq 1.7.\end{aligned}$$

□

Proof of Lemma 3.2(b): The proof is completely analogous to the proof of part (a). From (3.16) we have

$$\hat{I}(q) - \frac{1}{4}\delta_{1,q} = \int_0^{\theta^*} \hat{W}_q(\theta) u'(\cos\theta) \sin\theta \, d\theta + \int_{\theta^*}^{\pi/2} \hat{W}_q(\theta) u'(\cos\theta) \sin\theta \, d\theta - \frac{1}{2m},$$

where again $\theta^* \in [0, \frac{\pi}{2}]$ is defined such that $\cos\theta^* = x_m$ (see Proposition 3.4). Using Propostion 3.6 with the corresponding choice of θ_q we obtain the lower estimate

$$\hat{I}(q) - \frac{1}{4}\delta_{1,q} \geq \int_{\theta^*}^{\pi/2} \hat{W}_q(\theta)u'(\cos\theta)\sin\theta\, d\theta - \frac{1}{2m} \geq 2.36\, u(\cos\theta^*) - \frac{1}{2m}$$
$$\geq -\frac{2.18}{2m}.$$

In order to obtain the upper estimate we distinguish two cases. For $\theta^* \leq \theta_q$ we have

$$\hat{I}(q) - \frac{1}{4}\delta_{1,q} \leq \hat{W}_q(\theta^*)\int_0^{\theta^*} u'(\cos\theta)\sin\theta d\theta + \hat{W}_q(\theta^*)\int_{\theta^*}^{\pi/2} u'(\cos\theta)\sin\theta\, d\theta - \frac{1}{2m}$$
$$= \hat{W}_q(\theta^*)\frac{1}{2m} - \frac{1}{2m} \leq \frac{0.7}{2m} \leq \frac{2.18}{2m}.$$

For $\theta^* \geq \theta_q$ we have

$$\hat{I}(q) - \frac{1}{4}\delta_{1,q} \leq -2.36\, u(\cos\theta)\Big|_0^{\theta^*} - 1.7\, u(\cos\theta)\Big|_{\theta^*}^{\pi/2} - \frac{1}{2m}$$
$$= u(x_m)(1.7 - 2.36) + \frac{2.36}{2m} - \frac{1}{2m} \leq \frac{0.66}{4m} + \frac{1.36}{2m} = \frac{1.69}{2m} \leq \frac{2.18}{2m}.$$

This completes the proof of Lemma 3.2. □

3.3. Proof of Lemma 3.3

3.3.1. Part (a) of Lemma 3.3

We start by introducing the convenient notation $d_k = \sum_{j=k+1}^{m-1} c_j$ for $k = 0, \ldots, m-1, d_{m-1} \equiv 0$ (cf. (2.22)). We state the following technical Proposition.

Proposition 3.7. *For all $m \geq 2$,*

$$c_1 = \frac{2m-2}{2m-1} < 1, \quad \text{and} \quad \gamma \equiv 1 - \frac{c_1}{4} > \frac{3}{4}, \tag{3.17}$$

$$\sum_{j=0}^{m-1} d_j = \frac{m}{2}c_1, \tag{3.18}$$

$$\frac{1}{2}\sqrt{m\pi} - 1 \leq d_0 \leq \frac{1}{2}\sqrt{m\pi}. \tag{3.19}$$

Proof: By definition, we have

$$c_1 = \frac{2^{2-2m}}{A_m}\binom{2m-2}{m-2} = 2^{2-2m}\frac{(2m-2)!}{(m-2)!m!}\prod_{j=1}^{m}\frac{2j}{2j-1}.$$

Now, since

$$\frac{(2m-2)!}{\prod_{j=1}^{m}(2j-1)} = 2^{m-1}\frac{(m-1)!}{2m-1}, \quad \text{and} \quad \frac{\prod_{j=1}^{m}2j}{m!} = 2^m,$$

we obtain $c_1 = \frac{2m-2}{2m-1} < 1$ and hence $\gamma > \frac{3}{4}$. Hence the first part of the Proposition is proved.

In order to prove the second part, we observe that by definition $\sum_{j=0}^{m-1} d_j = \sum_{j=1}^{m-1} jc_j$. This implies that,

$$\sum_{j=0}^{m-1} d_j = \frac{2^{2-2m}}{A_m}\sum_{k=0}^{m-2}(m-1-k)\binom{2m-2}{k}.$$

Since

$$(k+1)\binom{2m-2}{k+1} - k\binom{2m-2}{k} = 2(m-1-k)\binom{2m-2}{k},$$

we arrive at

$$\begin{aligned}\sum_{j=0}^{m-1} d_j &= \frac{2^{2-2m}}{2A_m}\sum_{k=0}^{m-2}\left[(k+1)\binom{2m-2}{k+1} - k\binom{2m-2}{k}\right]\\ &= \frac{2^{2-2m}}{2A_m}(m-1)\binom{2m-2}{m-1} = \frac{m}{2}\frac{2^{2-2m}}{A_m}\binom{2m-2}{m-2} = \frac{m}{2}c_1.\end{aligned}$$

This proves the second part of the Proposition.

It now remains to prove the last part of the Proposition. First, we will derive a convenient expression for d_0. Since $\sum_{j=0}^{2m-2}\binom{2m-2}{j} = 2^{2m-2}$ we have

$$\begin{aligned}d_0 &= \frac{2^{2-2m}}{A_m}\sum_{k=0}^{m-2}\binom{2m-2}{k} = \frac{2^{2-2m}}{2A_m}\left[2^{2m-2} - \binom{2m-2}{m-1}\right]\\ &= \frac{1}{2A_m} - \frac{2^{2-2m}}{2A_m}\binom{2m-2}{m-1} = \frac{1}{2A_m} - \frac{1}{2}\frac{m}{m-1}c_1.\end{aligned}$$

Using the definition of A_m we obtain

$$d_0 = \frac{\sqrt{\pi}}{2}\frac{\Gamma(m+1)}{\Gamma(m+1/2)} - \frac{m}{2m-1}. \tag{3.20}$$

Next, we note the following estimate for the quotient of Gamma functions

$$0 \le \ln \Gamma(z) - \left(z - \frac{1}{2}\right) \ln z + z - \frac{1}{2} \ln(2\pi) \le \frac{1}{12z}, \tag{3.21}$$

for $z > 1$ (see e.g. Ref. 1, (6.1.42)). Thus

$$\ln \Gamma(m+1) \ge \left(m + \frac{1}{2}\right) \ln(m+1) - (m+1) + \frac{1}{2} \ln(2\pi)$$

$$\ln \Gamma\left(m + \frac{1}{2}\right) \le m \ln\left(m + \frac{1}{2}\right) - \left(m + \frac{1}{2}\right) + \frac{1}{2} \ln(2\pi) + \frac{1}{12(m + \frac{1}{2})},$$

so that

$$\ln \Gamma(m+1) - \ln \Gamma\left(m + \frac{1}{2}\right) \ge \frac{1}{2} \ln m + m \ln\left(\frac{m+1}{m + \frac{1}{2}}\right) - \frac{1}{2} - \frac{1}{12m}.$$

Further, since

$$\ln\left(\frac{m+1}{m + \frac{1}{2}}\right) \ge \frac{1}{2m+1} - \frac{1}{2(2m+1)^2},$$

we arrive at

$$\ln \Gamma(m+1) - \ln \Gamma\left(m + \frac{1}{2}\right) \ge \frac{1}{2} \ln m + \frac{m}{2m+1} - \frac{1}{2} - \frac{m}{2(2m+1)^2} - \frac{1}{12m}$$
$$\ge \frac{1}{2} \ln m - \frac{1}{2m}.$$

Therefore,

$$\frac{\Gamma(m+1)}{\Gamma(m + \frac{1}{2})} \ge \sqrt{m} e^{-\frac{1}{2m}} \ge \sqrt{m}\left(1 - \frac{1}{2m}\right) = \sqrt{m} - \frac{1}{2\sqrt{m}}.$$

Inserting this inequality into (3.20) we then have

$$d_0 \ge \frac{\sqrt{\pi m}}{2} - \frac{\sqrt{\pi}}{4\sqrt{m}} - \frac{m}{2m-1} \ge \frac{\sqrt{\pi m}}{2} - 1, \quad \text{for } m \ge 2.$$

In order to prove the upper bound we deduce from (3.21) that

$$\ln \Gamma(m+1) - \ln \Gamma\left(m + \frac{1}{2}\right) \le \frac{1}{2} \ln(m+1) + m \ln\left(\frac{m+1}{m + \frac{1}{2}}\right) - \frac{1}{2} + \frac{1}{12m}.$$

Using

$$\ln\left(\frac{m+1}{m + \frac{1}{2}}\right) \le \frac{1}{2m+1} \quad \text{and} \quad \ln(m+1) \le \ln m + \frac{1}{m}$$

we obtain for $m \geq 2$

$$\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})} \leq \sqrt{m}e^{\frac{2}{5m}}.$$

The claim then follows from (3.20) and from the inequalities

$$\frac{\sqrt{\pi m}}{2}(e^{\frac{2}{5m}}-1) - \frac{m}{2m-1} \leq \frac{1}{2}\left(\sqrt{\pi m}e^{\frac{1}{5}}\frac{2}{5m}-1\right) \leq \frac{1}{2}\left(\sqrt{\frac{\pi}{2}}e^{\frac{1}{5}}\frac{2}{5}-1\right) < 0$$

for $m \geq 2$. □

The next result will be used in the proofs of all parts of Lemma 3.3.

Proposition 3.8. *The following exact relation holds,*

$$\|\hat{Q}\|_{\infty\to 1} = \frac{d_0^2}{4\gamma} + \frac{m}{2}c_1. \tag{3.22}$$

Proof: A straightforward calculation, using (3.18), shows that

$$\begin{aligned}\|\hat{Q}\|_{\infty\to 1} &= \frac{1}{\gamma}\left(d_0 + d_1 + \frac{1}{4}d_1^2 + \gamma\sum_{j=2}^{m-1} d_j\right)\\ &= \frac{1}{\gamma}\left((1-\gamma)(d_0+d_1) + \frac{1}{4}d_1^2 + \gamma\frac{m}{2}c_1\right).\end{aligned}$$

The result then follows from the facts that $1-\gamma = \frac{c_1}{4}$ and $d_1 = d_0 - c_1$. □

Proof of Lemma 3.3(a): The first part of the Lemma follows easily from (3.22), (3.17) and (3.19). □

3.3.2. Parts (b) and (c) of Lemma 3.3

For convenience, we will write the $(m-1)$-vector v as a sum of two vectors $v = v^0 + v^1$ with v^0 and v^1 given by,

$$v^0 = \left[\frac{1}{2}\sqrt{\frac{m}{2m-1}} - \frac{1}{2\sqrt{m}}, -\frac{1}{2\sqrt{m}}, \dots, -\frac{1}{2\sqrt{m}}\right], \tag{3.23}$$

and

$$v^1 = \sqrt{\frac{m}{2m-1}}\left[I(1) - \frac{1}{2}, I(2), \dots, I(m-1)\right]. \tag{3.24}$$

The main feature of this splitting is that the entries of v^0 do not depend on the I-functions and that, by Lemma 3.2, the entries of v^1 can be estimated by

$$|v_j^1| \leq \frac{D}{2m}\sqrt{\frac{m}{2m-1}}, \quad \text{for all } j = 1, \dots, m-1. \tag{3.25}$$

Recalling that $\hat{Q}$ is symmetric, it is straightforward to check that we have the following estimates on $\|v\hat{Q}\|_1$ and $v\hat{Q}v^t$:

$$\|v\hat{Q}\|_1 \leq \|v^0\hat{Q}\|_1 + \frac{D}{2m}\sqrt{\frac{m}{2m-1}}\|\hat{Q}\|_{\infty\to 1}, \tag{3.26}$$

$$v\hat{Q}v^t \leq v^0\hat{Q}(v^0)^t + \frac{D}{m}\sqrt{\frac{m}{2m-1}}\|v^0\hat{Q}\|_1 + \frac{D^2}{4m(2m-1)}\|\hat{Q}\|_{\infty\to 1}. \tag{3.27}$$

It will turn out that we need to prove parts (b) and (c) of Lemma 3.3 in two steps. First, we consider the case $2 \leq m \leq 32$ and we let Maple explicitly calculate the right hand sides of the above estimates. We then need explicit expressions for $\|v^0\hat{Q}\|_1$ and $v^0\hat{Q}(v^0)^t$ (recall that we already have an explicit expression for $\|\hat{Q}\|_{\infty\to 1}$). For the proof in the case $m \geq 33$ we will determine estimates for the right hand sides of (3.26) and (3.27). In particular we need to determine estimates on $\|v^0\hat{Q}\|_1$ and $v^0\hat{Q}(v^0)^t$. In order to get a good estimate on $\|v^0\hat{Q}\|_1$ we will use the following Proposition.

Proposition 3.9. *For $j = 1, \dots, m-1$,*

$$0 \leq \frac{d_j}{c_j} \leq \frac{d_1}{c_1}.$$

Proof: Define $a_j = \frac{d_{m-j}}{c_{m-j}}$, for $j = 1, \dots, m-1$. Since $c_{m-j} = c_{m-j+1}\frac{2m-j}{j-1}$ for $j \geq 2$ we have the recursion relation,

$$a_j = \frac{j-1}{2m-j}(a_{j-1}+1), \quad \text{for } 2 \leq j \leq m-1; \qquad a_1 = 0.$$

We now prove that a_j is increasing, which proves the Proposition. We prove by induction that $a_j \leq a_{j+1}$. For $j = 1$ this is obvious. Next, suppose that it is true for j. Then

$$\begin{aligned} a_{j+1} &= (a_j+1)\frac{j}{2m-j-1} \leq (a_{j+1}+1)\frac{j}{2m-j-1} \\ &= a_{j+2}\frac{2m-j-2}{2m-j-1}\frac{j}{j+1} \leq a_{j+2} \end{aligned}$$

which completes the proof. □

Proposition 3.10. *For $m \geq 2$,*

$$\|v^0 \hat{Q}\|_1 = \sum_{j=1}^{m-1} \frac{c_j}{2\gamma} \left| \frac{1}{\sqrt{m}} \left(1 + \frac{d_1}{4} + \gamma \frac{d_j}{c_j}\right) - \sqrt{\frac{m}{2m-1}} \right|, \tag{3.28}$$

$$\|v^0 \hat{Q}\|_1 \leq 0.2869\sqrt{m}. \tag{3.29}$$

Proof: A straightforward calculation using the fact that $\frac{d_0}{c_1} = 1 + \frac{d_1}{4} + \gamma \frac{d_1}{c_1}$ shows that the j-th entry of $v^0 \hat{Q}$ is given by

$$(v^0 \hat{Q})_j = \frac{c_j}{2\gamma} \left(\sqrt{\frac{m}{2m-1}} - \frac{1}{\sqrt{m}} \left(1 + \frac{d_1}{4} + \gamma \frac{d_j}{c_j}\right) \right). \tag{3.30}$$

This proves the first part of the Proposition. In order to prove the second part we obtain an estimate for the absolute value term in (3.28). For all $m \geq 2$ we have by (3.19) and Proposition 3.9 that

$$\begin{aligned} \frac{1}{\sqrt{m}} \left(1 + \frac{d_1}{4} + \gamma \frac{d_j}{c_j}\right) - \sqrt{\frac{m}{2m-1}} &\leq \frac{1}{\sqrt{m}} \left(1 + \frac{d_1}{4} + \gamma \frac{d_1}{c_1}\right) - \sqrt{\frac{m}{2m-1}} \\ &= \frac{1}{\sqrt{m}} \frac{d_0}{c_1} - \sqrt{\frac{m}{2m-1}} \\ &\leq \begin{cases} \frac{\sqrt{\pi}}{2} \frac{2m-1}{2m-2} - \frac{1}{\sqrt{2}} \leq 0.41, & \text{for } m \geq 3. \\ 0, & \text{for } m = 2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\sqrt{m}} \left(1 + \frac{d_1}{4} + \gamma \frac{d_j}{c_j}\right) - \sqrt{\frac{m}{2m-1}} &\geq \frac{1}{\sqrt{m}} \left(1 + \frac{d_1}{4}\right) - \sqrt{\frac{m}{2m-1}} \\ &= \frac{1}{\sqrt{m}} \left(\gamma + \frac{d_0}{4}\right) - \sqrt{\frac{m}{2m-1}} \\ &\geq \frac{\sqrt{\pi}}{8} - \sqrt{\frac{m}{2m-1}} + \frac{1}{2\sqrt{m}} \geq \frac{\sqrt{\pi}}{8} - \frac{1}{\sqrt{2}}. \end{aligned}$$

This then implies by (3.28) that,

$$\|v^0\hat{Q}\|_1 \leq \frac{d_0}{2\gamma}\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{\pi}}{8}\right) \leq \frac{\sqrt{\pi}}{3}\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{\pi}}{8}\right)\sqrt{m}$$
$$\leq 0.2869\sqrt{m}, \quad \text{for } m \geq 2, \tag{3.31}$$

and the Proposition is proved. □

Proposition 3.11. *For $m \geq 2$,*

$$v^0\hat{Q}(v^0)^t = -\frac{1}{2\gamma}\frac{d_0}{\sqrt{m}}\sqrt{\frac{m}{2m-1}} + \frac{1}{2\gamma}\frac{m(m-1)}{(2m-1)^2} + \frac{c_1}{8} + \frac{d_0^2}{16\gamma m}. \tag{3.32}$$

Further,

$$v^0\hat{Q}(v^0)^t \leq \frac{0.246}{\sqrt{2m-1}}, \quad \textit{for } m \geq 33. \tag{3.33}$$

Proof: From (3.30), (3.23) and the fact that $\frac{d_0}{c_1} = 1 + \frac{d_1}{4} + \gamma\frac{d_1}{c_1}$ it follows that

$$v^0\hat{Q}(v^0)^t = \sum_{j=1}^{m-1}\frac{c_j}{2\gamma}\left(\sqrt{\frac{m}{2m-1}} - \frac{1}{\sqrt{m}}\left(1 + \frac{d_1}{4} + \gamma\frac{d_j}{c_j}\right)\right)v_j^0$$
$$= \frac{c_1}{2\gamma}\left(\sqrt{\frac{m}{2m-1}} - \frac{1}{\sqrt{m}}\frac{d_0}{c_1}\right)\frac{1}{2}\sqrt{\frac{m}{2m-1}}$$
$$- \frac{1}{4\gamma}\frac{1}{\sqrt{m}}\sum_{j=1}^{m-1}\left(c_j\sqrt{\frac{m}{2m-1}} - \frac{1}{\sqrt{m}}\left(c_j + c_j\frac{d_1}{4} + \gamma d_j\right)\right).$$

Now, from (3.18) and from the fact that $1 - \gamma = \frac{c_1}{4}$ we have

$$\sum_{j=1}^{m-1}\left(c_j + c_j\frac{d_1}{4} + \gamma d_j\right) = d_0 + \frac{1}{4}d_0 d_1 + \gamma\left(\frac{m}{2}c_1 - d_0\right)$$
$$= \gamma\frac{m}{2}c_1 + d_0\left(1 + \frac{d_1}{4} - \gamma\right) = \gamma\frac{m}{2}c_1 + \frac{d_0^2}{4}.$$

We obtain

$$v^0\hat{Q}(v^0)^t = \frac{c_1}{4\gamma}\frac{m}{2m-1} - \frac{1}{2\gamma}\frac{d_0}{\sqrt{m}}\sqrt{\frac{m}{2m-1}} + \frac{1}{4\gamma m}\left(\gamma\frac{m}{2}c_1 + \frac{d_0^2}{4}\right).$$

The first part of the Proposition then follows from (3.17). Next, from (3.32), (3.17), (3.19) and from the fact that $\frac{m(m-1)}{(2m-1)^2} < \frac{1}{4}$ we have

$$\begin{aligned} v^0 \hat{Q}(v^0)^t - \frac{0.246}{\sqrt{2m-1}} &\leq -\frac{\sqrt{\pi}}{4\gamma}\sqrt{\frac{m}{2m-1}} + \frac{1}{2\gamma\sqrt{2m-1}} + \frac{1}{6} + \frac{1}{8} + \frac{\pi}{64\gamma} \\ &\quad - \frac{0.246}{\sqrt{2m-1}} \\ &\leq -\frac{\sqrt{\pi}}{4\sqrt{2}\gamma} + \frac{7}{24} + \frac{\pi}{48} + \frac{2/3 - 0.246}{\sqrt{2m-1}} \\ &< 0, \quad \text{for } m \geq 33. \end{aligned}$$

In the last inequality we have used the fact that $\gamma \leq 0.754$ for $m \geq 33$. □

Proof of Lemma 3.4(b) and (c): First, consider the case $2 \leq m \leq 32$. From (3.26), (3.28) and (3.22) we obtain,

$$\begin{aligned} \|v\hat{Q}\|_1 \leq \sum_{j=1}^{m-1} \frac{c_j}{2\gamma} \left| \frac{1}{\sqrt{m}} \left(1 + \frac{d_1}{4} + \gamma \frac{d_j}{c_j}\right) - \sqrt{\frac{m}{2m-1}} \right| \\ + \frac{D}{2m}\sqrt{\frac{m}{2m-1}} \left(\frac{d_0^2}{4\gamma} + \frac{m}{2}c_1\right), \end{aligned}$$

and from (3.27), (3.32), (3.29) and (3.22) we obtain

$$\begin{aligned} v\hat{Q}v^t - \frac{1}{\sqrt{2m-1}} &\leq -\frac{d_0}{2\gamma}\frac{1}{\sqrt{2m-1}} + \frac{1}{2\gamma}\frac{m(m-1)}{(2m-1)^2} + \frac{c_1}{8} + \frac{d_0^2}{16\gamma m} \\ &\quad + (0.2869D - 1)\frac{1}{\sqrt{2m-1}} + \frac{D^2}{4m(2m-1)}\left(\frac{d_0^2}{4\gamma} + \frac{m}{2}c_1\right). \end{aligned}$$

We now let Maple calculate explicitly the right hand sides of these estimates for $2 \leq m \leq 32$, and we see that the Lemma is indeed satisfied in this case.

Next, we consider the case $m \geq 33$. From Eqs. (3.26) and (3.29) and from Lemma 3.3(a) we have

$$\|v\hat{Q}\|_1 \leq \left[0.2869 + \frac{D}{2\sqrt{2m-1}}\left(\frac{\pi}{12} + \frac{1}{2}\right)\right]\sqrt{m} \leq 0.3918\sqrt{m}, \quad \text{for } m \geq 33.$$

Further, from (3.27), (3.29), (3.33) and Lemma 3.4(a) it follows that

$$v\hat{Q}v^t \leq \left[0.246 + 0.2869D + \frac{D^2}{4\sqrt{2m-1}}\left(\frac{\pi}{12} + \frac{1}{2}\right)\right]\frac{1}{\sqrt{2m-1}}$$
$$< \frac{1}{\sqrt{2m-1}}, \quad \text{for } m \geq 33.$$

This concludes the proof of Lemma 3.4. □

4. ASYMPTOTICS OF ϕ_n, ψ_1 AND ψ_2 ON THE POSITIVE REAL LINE

The goal of this section is to derive the leading order behavior and error bounds for the functions ϕ_n, ψ_1 and ψ_2 which appear in the basis of $\mathfrak{g}_1$, $\mathfrak{g}_2$ (see Lemma 2.2). These results are stated in Lemmas 4.8–4.12 below. They will be used in the subsequent Sec. 5 to determine the asymptotic behavior of the matrix B defined by (2.27). We present our results for the rescaled functions

$$\hat{\phi}_n(x) = \sqrt{\beta_n}\phi_n(\beta_n x), \quad \hat{\psi}_r(x) = \sqrt{\beta_n}\psi_r(\beta_n x), \quad r = 1, 2, \tag{4.1}$$

where β_n denotes the Mhaskar–Rakhmanov–Saff number (see Sec. 4.1 below). In this rescaling all zeros of $\hat{\phi}_n$ lie in the interval [0, 1].

As is well-known in the theory of classical orthogonal polynomials, there are different asymptotic descriptions of the orthogonal polynomials in different parts of the complex plane. For our purposes it will suffice to consider $\hat{\phi}_n$, $\hat{\psi}_1$, $\hat{\psi}_2$ on $\mathbb{R}_+$. We find it most convenient for the analysis of Sec. 5 to split $(0, \infty)$ into four regions $(0, n^{-1}]$, $[n^{-1}, 1 - n^{\kappa-\frac{2}{3}}]$, $[1 - n^{\kappa-\frac{2}{3}}, 1 + n^{\kappa-\frac{2}{3}}]$ and $[1 + n^{\kappa-\frac{2}{3}}, \infty)$, which are called the Bessel-, bulk-, Airy- and exponential regions, respectively. Here, κ could be any sufficiently small positive constant. To be definite we choose once and for all,

$$\kappa = \frac{1}{12}. \tag{4.2}$$

The results of this section are corollaries of Ref. 23, where the asymptotic behavior of orthogonal polynomials of Laguerre type has been derived. For the convenience of the reader we summarize the relevant results from Ref. 23 in Sec. 4.1. After some auxiliary considerations in Sec. 4.2 we then derive the asymptotic description for $\hat{\phi}_n$ in Sec. 4.3 (Lemma 4.8) and for $\hat{\psi}_r$ ($r = 1, 2$) in Sec. 4.4 (Lemmas 4.9–4.12).

4.1. Relevant Results from Ref. 23

In order to describe the asymptotics of the functions $\hat{\phi}_n$ and $\hat{\psi}_r$ ($r = 1, 2$) on $\mathbb{R}_+$ we first introduce the sequence of Mhaskar–Rakhmanov–Saff numbers, which we denote by β_n. For V as in (1.6), these numbers are uniquely determined for n sufficiently large by the equation, cf. (Ref. 23, (2.1))

$$\frac{1}{2\pi}\int_0^{\beta_n} V'(x)\sqrt{\frac{x}{\beta_n - x}}dx = n, \tag{4.3}$$

and they have a convergent power series expansion of the form, cf. (Ref. 23, Proposition 3.4)

$$\beta_n = n^{1/m}\sum_{k=0}^{\infty}\beta^{(k)}n^{-k/m}, \quad \beta^{(0)} = \left(\frac{1}{2}mq_m A_m\right)^{-1/m}, \quad A_m = \prod_{j=1}^{m}\frac{2j-1}{2j}. \tag{4.4}$$

Next, we introduce the equilibrium measure μ_n on $[0, \infty)$ in the presence of the rescaled external field $V_n(x) = \frac{1}{n}V(\beta_n x)$. This measure is absolutely continuous with respect to Lebesgue measure and its density ω_n is given by, cf. (Ref. 23, Proposition 3.12)

$$\omega_n(x) = \frac{d\mu_n}{dx}(x) = \frac{1}{2\pi}\sqrt{\frac{1-x}{x}}h_n(x)\chi_{(0,1]}, \tag{4.5}$$

where $h_n(x) = \sum_{k=0}^{m-1} h_{n,k}x^k$ is a real polynomial of degree $m-1$, and satisfies

$$\int_0^1 \sqrt{\frac{1-s}{s}}h_n(s)\,ds = 2\pi. \tag{4.6}$$

The coefficients $h_{n,k}$ can be expanded to any order in powers of $n^{-1/m}$. In particular, to any order $q = 1, 2, \ldots$, as $n \to \infty$, we have uniformly for x in compact sets

$$h_n(x) = h(x) + \sum_{k=1}^{q} h_{(k)}(x)n^{-k/m} + \mathcal{O}(n^{-(q+1)/m}), \tag{4.7}$$

where h is given by (2.34), cf. (Ref. 23, Proposition 3.9 and Remark 3.10). Furthermore, there exists a constant $h_0 > 0$ such that $h_n(x) \geq h_0$ for all n sufficiently large and $x \in [0, \infty)$, cf. [Ref. 23, Proposition 3.9].

Let f_n and $\tilde{f}_n$ be the biholomorpic maps (near 1 and 0, resp.) as defined in (Ref. 23, Remark 3.20) and (Ref. 23, Remark 3.26), respectively. These maps are of the form

$$f_n(x) = c_n n^{2/3}(x-1)\hat{f}_n(x), \quad \text{and} \quad \tilde{f}_n(x) = -\tilde{c}_n n^2 x\hat{\tilde{f}}_n(x), \tag{4.8}$$

where $\hat{f}_n$ and $\hat{\tilde{f}}_n$ are real analytic near 1 and 0, respectively, satisfying for n sufficiently large, cf. (Ref. 23, Remarks 3.20 and 3.26)

$$|\hat{f}_n(z) - 1| \leq C|z-1|, \quad \text{for } |z-1| \text{ small}, \tag{4.9}$$

$$|\hat{\tilde{f}}_n(z) - 1| \leq C|z|, \quad \text{for } |z| \text{ small}, \tag{4.10}$$

for some constant $C > 0$. The numbers c_n and $\tilde{c}_n$ are given by, cf. (Ref. 23, Remarks 3.20, 3.26 and 2.2)

$$c_n = \left(\frac{1}{2}h_n(1)\right)^{2/3} = \sum_{k=0}^{\infty} c^{(k)} n^{-k/m}, \quad c^{(0)} = (2m)^{2/3}, \tag{4.11}$$

$$\tilde{c}_n = \left(\frac{1}{2}h_n(0)\right)^{2} = \sum_{k=0}^{\infty} \tilde{c}^{(k)} n^{-k/m}, \quad \tilde{c}^{(0)} = \left(\frac{2m}{2m-1}\right)^{2}. \tag{4.12}$$

Further, we will need the conformal map φ from $\mathbb{C} \setminus [0,1]$ onto the exterior of the unit circle, cf. (Ref. 23, (2.11))

$$\varphi(z) = 2(z - 1/2) + 2z^{1/2}(z-1)^{1/2}, \quad \text{for } z \in \mathbb{C} \setminus [0,1].$$

For notational convenience, we also introduce for $z \in \mathbb{C} \setminus ((-\infty, 0] \cup [1,\infty))$ and $j = 1, 2$, the scalar functions, cf. (Ref. 23, (5.3) and (5.13))

$$\eta_j(z) = \frac{1}{2}(\alpha \pm 1) \arccos(2z - 1), \tag{4.13}$$

$$\zeta_j(z) = \eta_j(z) - \frac{\pi\alpha}{2}. \tag{4.14}$$

Here and below, the $+$ sign in $\pm$ holds for η_1 whereas the $-$ sign holds for η_2. The function $\arccos z$ is defined as the inverse function of $\cos z : \{0 < \operatorname{Re} z < \pi\} \to \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty))$. Further, introduce for $j = 1, 2$,

$$F_{n,j}(x) = -\frac{n}{2}\int_1^x \sqrt{\frac{1-s}{s}}\, h_n(s)\, ds + \eta_j(x) - \frac{\pi}{4}, \quad \text{for } x \in [0,1]. \tag{4.15}$$

Throughout the rest of this paper we denote $F_{n,1}$ by F_n for brevity.

Theorem 4.1. (Ref. 23, Theorem 2.4) *The functions $\hat{\phi}_n(x) = \sqrt{\beta_n}\phi_n(\beta_n x)$ have the following asymptotic behavior on the positive real line as $n \to \infty$. There exists $\delta > 0$ (sufficiently small) such that:*

(i) *Uniformly for* $x \in (0, \delta]$,

$$\hat{\phi}_n(x) = (-1)^n \frac{\sqrt{2}(-\tilde{f}_n(x))^{1/4}}{x^{1/4}(1-x)^{1/4}}[\sin\zeta_1(x)J_\alpha(2(-\tilde{f}_n(x))^{1/2})(1+\mathcal{O}(1/n)) + \cos\zeta_1(x)J'_\alpha(2(-\tilde{f}_n(x))^{1/2})(1+\mathcal{O}(1/n))]. \tag{4.16}$$

(ii) *Uniformly for* $x \in [\delta, 1-\delta]$,

$$\hat{\phi}_n(x) = \sqrt{\frac{2}{\pi}} \frac{\cos F_n(x)}{x^{1/4}(1-x)^{1/4}} + \mathcal{O}\left(\frac{1}{nx^{1/4}(1-x)^{1/4}}\right), \tag{4.17}$$

where $F_n = F_{n,1}$ *is defined by* (4.15).

(iii) *Uniformly for* $x \in [1-\delta, 1+\delta]$,

$$\hat{\phi}_n(x) = \frac{\sqrt{2}}{x^{1/4}}\left[\cos\eta_1(x)\left|\frac{f_n(x)}{x-1}\right|^{1/4} \mathrm{Ai}\,(f_n(x))(1+\mathcal{O}(1/n)) - \frac{\sin\eta_1(x)}{(1-x)^{1/2}}\left|\frac{f_n(x)}{x-1}\right|^{-1/4} \mathrm{Ai}\,'(f_n(x))(1+\mathcal{O}(1/n))\right]. \tag{4.18}$$

(iv) *Uniformly for* $x \in [1+\delta, \infty]$,

$$\hat{\phi}_n(x) = \frac{1}{\sqrt{2\pi}} \frac{\varphi(x)^{\frac{1}{2}(\alpha+1)}}{x^{1/4}(x-1)^{1/4}} \exp\left[-\frac{n}{2}\int_1^x \sqrt{\frac{s-1}{s}} h_n(s)ds\right](1+\mathcal{O}(1/n)) \tag{4.19}$$

Remark 4.2. Note that the functions η_j are only analytic in $\mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$. However, since $\eta_{j,+} = -\eta_{j,-}$ on $(1, \infty)$ the functions $\cos\eta_j(z)$ and $\frac{\sin\eta_j(z)}{(1-z)^{1/2}}$ are analytic near 1. Furthermore, the reader can verify that these functions have the following behavior near 1,

$$\cos\eta_{1,2}(x) = 1 + \mathcal{O}(x-1),$$
$$\frac{\sin\eta_{1,2}(x)}{(1-x)^{1/2}} = (\alpha \pm 1) + \mathcal{O}(x-1), \quad \text{as } x \to 1, \tag{4.20}$$

and using the fact that $\varphi(z) = e^{i\arccos(2z-1)}$ for $z \in \mathbb{C}_+$ one can verify that for $x > 1$,

$$\cos\eta_{1,2}(x) = \frac{1}{2}\left(\varphi(x)^{\frac{1}{2}(\alpha\pm1)} + \varphi(x)^{-\frac{1}{2}(\alpha\pm1)}\right), \tag{4.21}$$

$$\frac{\sin\eta_{1,2}(x)}{(1-x)^{1/2}} = \frac{1}{2\sqrt{x-1}}\left(\varphi(x)^{\frac{1}{2}(\alpha\pm1)} - \varphi(x)^{-\frac{1}{2}(\alpha\pm1)}\right). \tag{4.22}$$

For later reference we observe that

$$\zeta_{1,2}(z) = \pm\frac{\pi}{2} - (\alpha \pm 1)z^{1/2}(1 + \mathcal{O}(z)), \quad \text{as } z \to 0. \tag{4.23}$$

In order to obtain the asymptotics of the functions $\hat{\psi}_r$ $(r = 1, 2)$, see (4.1), we write them in terms of the RH problem for orthogonal polynomials due to Fokas, Its and Kitaev.[9] Let Y be the solution of the RH problem for orthogonal polynomials associated to the weight $x^\alpha e^{-V(x)}$ on $[0, \infty)$,

$$Y(z) = \begin{pmatrix} \frac{1}{\gamma_n} p_n(z) & \frac{1}{\gamma_n} C(p_n w)(z) \\ -2\pi i \gamma_{n-1} p_{n-1}(z) & -2\pi i \gamma_{n-1} C(p_{n-1} w)(z) \end{pmatrix}, \quad \text{for } z \in \mathbb{C} \setminus [0, \infty),$$

where $\gamma_n > 0$ is the leading coefficient of $p_n(z)$, $p_n(z) = \gamma_n z^n + \cdots$. Define a 2×2 matrix valued function U by

$$U(z) = \beta_n^{-(n+\frac{\alpha}{2})\sigma_3} Y(\beta_n z) \beta_n^{\frac{1}{2}\alpha\sigma_3}, \quad \text{for } z \in \mathbb{C} \setminus [0, \infty),$$

where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the third Pauli matrix, cf. (Ref. 23, (3.14)). Using Eqs. (2.11), (2.9), (2.10), together with the defining relation for the rescaled external field $V_n(x) = \frac{1}{n} V(\beta_n x)$, it is straightforward to verify that

$$\begin{aligned} \begin{pmatrix} \hat{\psi}_2(x) \\ \hat{\psi}_1(x) \end{pmatrix} &= \frac{n^{-1/2}}{x} \begin{pmatrix} 1 & 0 \\ 0 & \frac{i\alpha}{2\pi} \end{pmatrix} (-1/d_n)^{\sigma_3} Y(0)^{-1} Y(\beta_n x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\beta_n x)^{\frac{\alpha}{2}} e^{-\frac{1}{2}V(\beta_n x)} \\ &= \frac{n^{-1/2}}{x\sqrt{\pi}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{i\alpha}{2} \end{pmatrix} \left(-\frac{\sqrt{\pi}}{d_n} \beta_n^{\frac{1}{2}\alpha}\right)^{\sigma_3} U(0)^{-1} U(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} x^{\frac{\alpha}{2}} e^{-\frac{1}{2}nV_n(x)}. \end{aligned}$$

The constant matrix $U(0)^{-1}$ has been determined in (Ref. 23, Remark 5.5). Inserting this information and the defining relation

$$-\frac{1}{d_n} \equiv \frac{\tilde{c}_n^{\frac{\alpha}{2}} n^\alpha e^{\frac{1}{2}V(0)}}{\Gamma(\alpha)} \beta_n^{-\frac{1}{2}\alpha}, \tag{4.24}$$

into the previous equation, we obtain

$$\begin{aligned} \begin{pmatrix} \hat{\psi}_2(x) \\ \hat{\psi}_1(x) \end{pmatrix} &= (-1)^n \frac{n^{-1/2}}{x\sqrt{\pi}} \begin{pmatrix} \frac{\alpha}{4} & \frac{1}{2} \\ -\frac{\alpha}{4} & \frac{1}{2} \end{pmatrix} (\tilde{c}_n n^2)^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1-\alpha & -i(\alpha+1) \\ 1 & i \end{pmatrix} \\ &\quad \times 2^{\alpha\sigma_3} R(0)^{-1} e^{-\frac{1}{2}n\ell_n\sigma_3} U(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} x^{\frac{\alpha}{2}} e^{-\frac{1}{2}nV_n(x)}, \end{aligned} \tag{4.25}$$

where R is the result of the series of transformations $Y \mapsto U \mapsto T \mapsto S \mapsto R$ in the Deift–Zhou steepest-descent analysis of the RH problem for Y, see (Ref. 23, Sec. 3), and where ℓ_n is the Lagrange multiplier given in (Ref. 23, Proposition

3.12). The first column of U has been determined in (Ref. 23, Sec. 5), and in the next theorem we summarize its description on $\mathbb{R}_+$.

Theorem 4.3. *The first column of U has the following description on $\mathbb{R}_+$.*

(i) *[Ref. 23, (5.14)] For $x \in (0, \delta]$,*

$$U(x)\begin{pmatrix}1\\0\end{pmatrix} = x^{-\frac{\alpha}{2}} e^{\frac{1}{2}nV_n(x)} e^{\frac{1}{2}n\ell_n\sigma_3}(-1)^n \frac{\sqrt{\pi}(-\tilde{f}_n(x))^{1/4}}{x^{1/4}(1-x)^{1/4}} \times R(x)2^{-\alpha\sigma_3}\begin{pmatrix}\sin\zeta_1(x) & \cos\zeta_1(x)\\ -i\sin\zeta_2(x) & -i\cos\zeta_2(x)\end{pmatrix}\begin{pmatrix}J_\alpha(2(-\tilde{f}_n(x))^{1/2})\\ J_\alpha'(2(-\tilde{f}_n(x))^{1/2})\end{pmatrix}. \tag{4.26}$$

(ii) *[Ref. 23, (5.6)] For $x \in [\delta, 1-\delta]$,*

$$U(x)\begin{pmatrix}1\\0\end{pmatrix} = x^{-\frac{\alpha}{2}} e^{\frac{1}{2}nV_n(x)} e^{\frac{1}{2}n\ell_n\sigma_3} \frac{1}{x^{1/4}(1-x)^{1/4}} R(x)2^{-\alpha\sigma_3} \times \begin{pmatrix}\cos F_{n,1}(x)\\ -i\cos F_{n,2}(x)\end{pmatrix}, \tag{4.27}$$

where $F_{n,j}$ is defined by (4.15).

(iii) *[Ref. 23, (5.9)] For $x \in [1-\delta, 1+\delta]$,*

$$U(x)\begin{pmatrix}1\\0\end{pmatrix} = x^{-\frac{\alpha}{2}} e^{\frac{1}{2}nV_n(x)} e^{\frac{1}{2}n\ell_n\sigma_3} \frac{\sqrt{\pi}}{x^{1/4}} R(x)2^{-\alpha\sigma_3} \times \begin{pmatrix}\cos\eta_1(x) & -\frac{\sin\eta_1(x)}{(1-x)^{1/2}}\\ -i\cos\eta_2(x) & i\frac{\sin\eta_2(x)}{(1-x)^{1/2}}\end{pmatrix} \left|\frac{f_n(x)}{x-1}\right|^{\sigma_3/4} \begin{pmatrix}\mathrm{Ai}\,(f_n(x))\\ \mathrm{Ai}'(f_n(x))\end{pmatrix}. \tag{4.28}$$

(iv) *[Ref. 23, (5.4), see also (3.41) and (2.8)] For $x \in [1+\delta, \infty]$,*

$$U(x)\begin{pmatrix}1\\0\end{pmatrix} = x^{-\frac{\alpha}{2}} e^{\frac{1}{2}nV_n(x)} e^{\frac{1}{2}n\ell_n\sigma_3} \frac{1}{2x^{1/4}(x-1)^{1/4}} \times R(x)2^{-\alpha\sigma_3}\begin{pmatrix}\varphi(x)^{\frac{1}{2}(\alpha+1)}\\ -i\varphi(x)^{\frac{1}{2}(\alpha-1)}\end{pmatrix} \exp\left(-\frac{n}{2}\int_1^x \sqrt{\frac{s-1}{s}}h_n(s)\,ds\right). \tag{4.29}$$

4.2. Auxiliary Results

In order to determine the asymptotics of the functions $\hat{\phi}_n$, $\hat{\psi}_1$ and $\hat{\psi}_2$ on the positive real line we will make use of the following auxiliary results.

Proposition 4.4. *Let $j = 1, 2$. The following matching formulae hold.*

(i) *Uniformly for $x \in [\frac{1}{2}n^{-1}, \delta]$, as $n \to \infty$,*

$$(-\tilde{f}_n(x))^{1/4}[\sin \zeta_j(x) J_\alpha(2(-\tilde{f}_n(x))^{1/2}) + \cos \zeta_j(x) J'_\alpha(2(-\tilde{f}_n(x))^{1/2})]$$
$$= \frac{(-1)^n}{\sqrt{\pi}}(\cos F_{n,j}(x) + \tau_n(x) \sin F_{n,j}(x)) + \mathcal{O}(1/n), \tag{4.30}$$

with $\tau_n(x) = \frac{4\alpha^2 - 1}{16(-\tilde{f}_n(x))^{1/2}}$, and with $F_{n,j}$ given by (4.15).

(ii) *Uniformly for $x \in [1 - \delta, 1 - \frac{1}{2}n^{\kappa - \frac{2}{3}}]$, as $n \to \infty$,*

$$\cos \eta_j(x) |f_n(x)|^{1/4} \mathrm{Ai}\,(f_n(x)) - \sin \eta_j(x) |f_n(x)|^{-1/4} \mathrm{Ai}\,'(f_n(x))$$
$$= \frac{1}{\sqrt{\pi}} \cos F_{n,j}(x) + \mathcal{O}\left(\frac{1}{n(1-x)^{3/2}}\right). \tag{4.31}$$

(iii) *Uniformly for $x \in [1 + \frac{1}{2}n^{\kappa - \frac{2}{3}}, 1 + \delta]$, as $n \to \infty$,*

$$f_n(x)^{1/4} \mathrm{Ai}\,(f_n(x)) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{n}{2}\int_1^x \sqrt{\frac{s-1}{s}} h_n(s) ds\right)$$
$$\times (1 + \mathcal{O}(n^{-\frac{3}{2}\kappa})), \tag{4.32}$$

and

$$f_n(x)^{-1/4} \mathrm{Ai}\,'(f_n(x)) = -\frac{1}{2\sqrt{\pi}} \exp\left(-\frac{n}{2}\int_1^x \sqrt{\frac{s-1}{s}} h_n(s) ds\right)$$
$$\times (1 + \mathcal{O}(n^{-\frac{3}{2}\kappa})). \tag{4.33}$$

Proof: (i) From (4.8) and the fact that $\tilde{c}_n$ and $\hat{\tilde{f}}_n$ are positive, we have

$$2(-\tilde{f}_n(x))^{1/2} = i \lim_{z \to x + i0} 2\tilde{f}_n(z)^{1/2}, \quad \text{for } x \in (0, \delta].$$

Using in addition [Ref. 23, (2.10) and (2.8)], (4.6), (4.15) and the fact that $\eta_j = \zeta_j + \frac{\pi\alpha}{2}$ we arrive at,

$$2(-\tilde{f}_n(x))^{1/2} = \frac{n}{2}\int_0^x \sqrt{\frac{1-s}{s}}h_n(s)\,ds = \frac{n}{2}\int_1^x \sqrt{\frac{1-s}{s}}h_n(s)\,ds + \pi n$$
$$= -F_{n,j}(x) + \zeta_j(x) + \frac{\pi\alpha}{2} - \frac{\pi}{4} + \pi n, \quad \text{for } x \in (0,\delta].$$

By [Ref. 1, (9.2.5), (9.2.9) and (9.2.10)] this implies, uniformly for $x \in [\frac{1}{2}n^{-1}, \delta]$, as $n \to \infty$,

$$\begin{aligned}
&\sqrt{\pi}(-\tilde{f}_n(x))^{1/4} J_\alpha\left(2(-\tilde{f}_n(x))^{1/2}\right)\\
&= \cos\left(2(-\tilde{f}_n(x))^{1/2} - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) -\\
&\quad \tau_n(x)\sin\left(2(-\tilde{f}_n(x))^{1/2} - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{n^2 x}\right)\\
&= (-1)^n\left[-\sin(F_{n,j}(x) - \zeta_j(x)) + \tau_n(x)\cos(F_{n,j}(x) - \zeta_j(x)) + \mathcal{O}\left(\frac{1}{n}\right)\right],
\end{aligned} \tag{4.34}$$

and similarly by [Ref. 1, (9.2.11), (9.2.15) and (9.2.16)],

$$\begin{aligned}
&\sqrt{\pi}(-\tilde{f}_n(x))^{1/4} J_\alpha'(2(-\tilde{f}_n(x))^{1/2})\\
&= (-1)^n\left[\cos(F_{n,j}(x) - \zeta_j(x)) + \frac{4\alpha^2+3}{16(-\tilde{f}_n(x))^{1/2}}\right.\\
&\quad \left.\sin(F_{n,j}(x) - \zeta_j(x)) + \mathcal{O}\left(\frac{1}{n^2 x}\right)\right]\\
&= (-1)^n\left[\cos(F_{n,j}(x) - \zeta_j(x)) + \tau_n(x)\sin(F_{n,j}(x) - \zeta_j(x)) + \mathcal{O}\left(\frac{1}{n\sqrt{x}}\right)\right].
\end{aligned} \tag{4.35}$$

Together with the fact that $\cos\zeta_j(x) = \mathcal{O}(\sqrt{x})$ as $x \to 0$, which follows from (4.23), this yields (4.30).

(ii) From (4.8) and the fact that c_n and $\hat{f}_n$ are positive, we have

$$\frac{2}{3}(-f_n(x))^{2/3} = i\lim_{z\to x+i0}\frac{2}{3}f_n(z)^{3/2}, \quad \text{for } x \in [1-\delta, 1).$$

From [Ref. 23, (2.9) and (2.8)] and (4.15) we then obtain,

$$\frac{2}{3}(-f_n(x))^{3/2} = -\frac{n}{2}\int_1^x \sqrt{\frac{1-s}{s}}\,h_n(s)\,ds$$
$$= F_{n,j}(x) - \eta_j(x) + \frac{\pi}{4}, \quad \text{for } x \in [1-\delta, 1).$$

This implies by [Ref. 1, (10.4.60)], uniformly for $x \in [1-\delta, 1-\frac{1}{2}n^{\kappa-\frac{2}{3}}]$, as $n \to \infty$,

$$|f_n(x)|^{1/4}\mathrm{Ai}\,(f_n(x)) = \frac{1}{\sqrt{\pi}}\sin\left(\frac{2}{3}(-f_n(x))^{3/2} + \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{n(1-x)^{3/2}}\right)$$
$$= \frac{1}{\sqrt{\pi}}\cos(F_{n,j}(x) - \eta_j(x)) + \mathcal{O}\left(\frac{1}{n(1-x)^{3/2}}\right), \quad (4.36)$$

and similarly by [Ref. 1, (10.4.62)],

$$|f_n(x)|^{-1/4}\mathrm{Ai}'(f_n(x)) = \frac{1}{\sqrt{\pi}}\sin(F_{n,j}(x) - \eta_j(x)) + \mathcal{O}\left(\frac{1}{n(1-x)^{3/2}}\right). \quad (4.37)$$

After a straightforward calculation we obtain (4.31).

(iii) From [Ref. 23, (2.9) and (2.8)] we have

$$\frac{2}{3}f_n(x)^{3/2} = -\frac{n}{2}\int_x^1 \sqrt{\frac{s-1}{s}}\,h_n(s)\,ds, \quad \text{for } x \in (1, 1+\delta].$$

Using in addition [Ref. 1, (10.4.59) and (10.4.61)] it is simple to check that the last part of the Proposition is also satisfied. □

Proposition 4.5. *For every $L > 0$ we have as $n \to \infty$,*

$$J_\alpha\big(2(-\tilde{f}_n(x))^{1/2}\big) = J_\alpha\big(2\tilde{c}_n^{1/2}n\sqrt{x}\big) + \begin{cases} \mathcal{O}(n^\alpha x^{\frac{\alpha}{2}+1}), & \textit{uniformly for } x \in (0, Ln^{-2}], \\ \mathcal{O}(n^{1/2}x^{5/4}), & \textit{uniformly for } x \in [n^{-2}, 2n^{-1}], \end{cases} \quad (4.38)$$

$$J'_\alpha\big(2(-\tilde{f}_n(x))^{1/2}\big) = J'_\alpha\big(2\tilde{c}_n^{1/2}n\sqrt{x}\big) + \begin{cases} \mathcal{O}(n^{\alpha-1} x^{\frac{\alpha}{2}+\frac{1}{2}}), & \textit{uniformly for } x \in (0, Ln^{-2}], \\ \mathcal{O}(n^{1/2}x^{5/4}), & \textit{uniformly for } x \in [n^{-2}, 2n^{-1}]. \end{cases} \quad (4.39)$$

Proof: Note that

$$(-\tilde{f}_n(x))^{1/2} = \tilde{c}_n^{1/2} n\sqrt{x}(1+\mathcal{O}(x)), \quad \text{uniformly for } x \in (0, 2n^{-1}], \text{ as } n \to \infty.$$

Since $\sup_{y\in[0,C]} |y^{-(\alpha-1)} J'_\alpha(y)| < \infty$ for any $C > 0$, it is then simple to check that,

$$\begin{aligned} &J_\alpha\big(2(-\tilde{f}_n(x))^{1/2}\big) - J_\alpha\left(2\tilde{c}_n^{1/2} n\sqrt{x}\right) \\ &\quad = \left(2(-\tilde{f}_n(x))^{1/2} - 2\tilde{c}_n^{1/2} n\sqrt{x}\right)\int_0^1 J'_\alpha\left((1-t)2(-\tilde{f}_n(x))^{1/2} + 2t\tilde{c}_n^{1/2} n\sqrt{x}\right) dt \\ &\quad = \mathcal{O}(n^\alpha x^{\frac{\alpha}{2}+1}), \end{aligned}$$

uniformly for $x \in (0, Ln^{-2}]$, as $n \to \infty$. The determination of the error term in $[n^{-2}, 2n^{-1}]$ is analogous using $\sup_{y\in[C,\infty)} |\sqrt{y} J'_\alpha(y)| < \infty$ for any $C > 0$.

Similarly, using the facts $\sup_{y\in[0,C]} |y^{-(\alpha-2)} J''_\alpha(y)| < \infty$ and $\sup_{y\in[C,\infty)} |\sqrt{y} J''_\alpha(y)| < \infty$ for any $C > 0$, one proves (4.39). □

Corollary 4.6. *For every $L > 0$ we have as $n \to \infty$,*

$$J_\alpha(2(-\tilde{f}_n(x))^{1/2}) = \begin{cases} \mathcal{O}(n^\alpha x^{\frac{\alpha}{2}}), & \text{uniformly for } x \in (0, Ln^{-2}], \\ \mathcal{O}(n^{-1/2} x^{-1/4}), & \text{uniformly for } x \in [n^{-2}, 2n^{-1}]. \end{cases} \tag{4.40}$$

$$J'_\alpha(2(-\tilde{f}_n(x))^{1/2}) = \begin{cases} \mathcal{O}(n^{\alpha-1} x^{\frac{\alpha}{2}-\frac{1}{2}}), & \text{uniformly for } x \in (0, Ln^{-2}], \\ \mathcal{O}(n^{-1/2} x^{-1/4}), & \text{uniformly for } x \in [n^{-2}, 2n^{-1}]. \end{cases} \tag{4.41}$$

Proof: This follows from the facts

$$\sup_{y\in[0,C]} |y^{-\alpha} J_\alpha(y)| < \infty, \qquad \sup_{y\in[C,\infty)} |\sqrt{y} J_\alpha(y)| < \infty,$$

$$\sup_{y\in[0,C]} |y^{-(\alpha-1)} J'_\alpha(y)| < \infty, \qquad \sup_{y\in[C,\infty)} |\sqrt{y} J'_\alpha(y)| < \infty$$

for any $C > 0$. □

Proposition 4.7. *Uniformly for $x \in [1 - 2n^{\kappa-\frac{2}{3}}, 1 + 2n^{\kappa-\frac{2}{3}}]$, as $n \to \infty$,*

$$\left|\frac{f_n(x)}{x-1}\right|^{1/4} \mathrm{Ai}\,(f_n(x)) = c_n^{1/4} n^{1/6} \mathrm{Ai}\,(c_n n^{2/3}(x-1)) + \mathcal{O}(n^{-1/2+\frac{9}{4}\kappa}), \tag{4.42}$$

$$\left|\frac{f_n(x)}{x-1}\right|^{-1/4} \mathrm{Ai}'(f_n(x)) = \mathcal{O}(n^{-1/6+\frac{1}{4}\kappa}). \tag{4.43}$$

Proof: Note that as $n \to \infty$

$$f_n(x) = c_n n^{2/3}(x-1)(1+\mathcal{O}(n^{\kappa-\frac{2}{3}})), \tag{4.44}$$

uniformly for $x \in [1-2n^{\kappa-\frac{2}{3}}, 1+2n^{\kappa-\frac{2}{3}}]$. Together with $|\mathrm{Ai}'(\xi)| \le C(1+|\xi|)^{1/4}$ for $\xi \in \mathbb{R}$ and C some positive constant, one can then verify that

$$\begin{aligned}
&\mathrm{Ai}\,(f_n(x)) - \mathrm{Ai}\,(c_n n^{2/3}(x-1)) \\
&\quad = (f_n(x) - c_n n^{2/3}(x-1)) \int_0^1 \mathrm{Ai}'((1-t)f_n(x) + tc_n n^{2/3}(x-1))\,dt \\
&\quad = \mathcal{O}(n^{-2/3+\frac{9}{4}\kappa}).
\end{aligned} \tag{4.45}$$

Equation (4.42) now follows from this equation together with (4.44) and the fact that the Airy function is bounded on the real line.

From (4.44) and from the fact that $|\mathrm{Ai}'(\xi)| \le C(1+|\xi|)^{1/4}$ we have $\mathrm{Ai}'(f_n(x)) = \mathcal{O}(n^{\frac{1}{4}\kappa})$. Together with (4.44) this proves Eq. (4.43). □

4.3. Asymptotic Behavior of $\hat{\phi}_n$

The asymptotic behavior of $\hat{\phi}_n$ on the positive real line is now given by the following Lemma.

Lemma 4.8. *The functions $\hat{\phi}_n(x) = \sqrt{\beta_n}\phi_n(\beta_n x)$ have the following asymptotic behavior on the positive real line, as $n \to \infty$.*

(i) *Bessel region: For every $L > 0$,*

$$\hat{\phi}_n(x) = \begin{cases} \mathcal{O}(n^{\alpha+\frac{1}{2}} x^{\frac{\alpha}{2}}), & \text{uniformly for } x \in (0, Ln^{-2}], \\ \mathcal{O}(x^{-1/4}), & \text{uniformly for } x \in [n^{-2}, 2n^{-1}]. \end{cases} \tag{4.46}$$

(ii) *Bulk region:*

$$\hat{\phi}_n(x) = \sqrt{\frac{2}{\pi}} \frac{\cos F_n(x)}{x^{1/4}(1-x)^{1/4}} + \mathcal{O}\left(\frac{1}{nx^{3/4}(1-x)^{7/4}}\right), \tag{4.47}$$

uniformly for $x \in [\frac{1}{2}n^{-1}, 1-\frac{1}{2}n^{\kappa-\frac{2}{3}}]$.

(iii) *Airy region:*

$$\hat{\phi}_n(x) = \sqrt{2}c_n^{1/4} n^{1/6} \mathrm{Ai}\,(c_n n^{2/3}(x-1)) + \mathcal{O}(n^{-1/6+\frac{1}{4}\kappa}), \tag{4.48}$$

uniformly for $x \in [1-2n^{\kappa-\frac{2}{3}}, 1+2n^{\kappa-\frac{2}{3}}]$.

(iv) *Exponential region: there exists a constant* $c > 0$ *such that,*

$$\hat{\phi}_n(x) = \mathcal{O}\big(e^{-c(x-1)n^{2/3}}\big), \quad \textit{uniformly for } x \in [1 + \tfrac{1}{2}n^{\kappa-\frac{2}{3}}, \infty). \tag{4.49}$$

Proof: (i) Using Eq. (4.16), Corollary 4.6 and the facts that $(-\tilde{f}_n(x))^{1/4} = \mathcal{O}(n^{1/2}x^{1/4})$, as $n \to \infty$, and $\cos\zeta_1(x) = \mathcal{O}(x^{1/2})$ as $x \to 0$, we obtain (4.46).

(ii) By (4.34) and (4.35),

$$(-\tilde{f}_n(x))^{1/4} J_\alpha(2(-\tilde{f}_n(x))^{1/4}) = \mathcal{O}(1), \quad (-\tilde{f}_n(x))^{1/4} J_\alpha'(2(-\tilde{f}_n(x))^{1/4}) = \mathcal{O}(1),$$

as $n \to \infty$, uniformly for $x \in [\frac{1}{2}n^{-1}, \delta]$. From (4.16), (4.30), and the estimate $\tau_n(x) = \mathcal{O}(\frac{1}{n\sqrt{x}})$, we then obtain,

$$\begin{aligned}\hat{\phi}_n(x) &= \frac{(-1)^n\sqrt{2}}{x^{1/4}(1-x)^{1/4}}[(-\tilde{f}_n(x))^{1/4}\sin\zeta_1(x) J_\alpha(2(-\tilde{f}_n(x))^{1/2}) \\ &\quad + (-\tilde{f}_n(x))^{1/4}\cos\zeta_1(x) J_\alpha'(2(-\tilde{f}_n(x))^{1/2}) + \mathcal{O}(1/n)] \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{x^{1/4}(1-x)^{1/4}}\left[\cos F_n(x) + \mathcal{O}\left(\frac{1}{n\sqrt{x}}\right)\right],\end{aligned} \tag{4.50}$$

as $n \to \infty$, uniformly for $x \in [\frac{1}{2}n^{-1}, \delta]$, where we recall that that $F_{n,1} \equiv F_n$. Further, from (4.36) and (4.37), we have

$$|f_n(x)|^{1/4}\mathrm{Ai}\,(f_n(x)) = \mathcal{O}(1), \quad |f_n(x)|^{-1/4}\mathrm{Ai}\,'(f_n(x)) = \mathcal{O}(1),$$

as $n \to \infty$, uniformly for $x \in [1-\delta, 1-n^{\kappa-\frac{2}{3}}]$. By (4.18) and (4.31) we then obtain

$$\begin{aligned}\hat{\phi}_n(x) &= \frac{\sqrt{2}}{x^{1/4}(1-x)^{1/4}}\Bigg[\cos\eta_1(x)|f_n(x)|^{1/4}\mathrm{Ai}\,(f_n(x)) \\ &\quad - \sin\eta_1(x)|f_n(x)|^{-1/4}\mathrm{Ai}\,'(f_n(x)) + \mathcal{O}\left(\frac{1}{n(1-x)^{3/2}}\right)\Bigg] \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{x^{1/4}(1-x)^{1/4}}\left[\cos F_n(x) + \mathcal{O}\left(\frac{1}{n(1-x)^{3/2}}\right)\right],\end{aligned} \tag{4.51}$$

as $n \to \infty$, uniformly for $x \in [1-\delta, 1-n^{\kappa-\frac{2}{3}}]$. Equations (4.17), (4.50) and (4.51) then yield (4.47).

(iii) Now, we prove the third part of the Proposition. From Eqs. (4.18) and (4.20) it follows readily that,

$$\hat{\phi}_n(x) = \sqrt{2}\left|\frac{f_n(x)}{x-1}\right|^{1/4} \mathrm{Ai}\,(f_n(x))(1+\mathcal{O}(n^{\kappa-\frac{2}{3}}))$$
$$-\sqrt{2}(\alpha+1)\left|\frac{f_n(x)}{x-1}\right|^{-1/4} \mathrm{Ai}'(f_n(x))(1+\mathcal{O}(n^{\kappa-\frac{2}{3}})), \qquad (4.52)$$

as $n \to \infty$, uniformly for $x \in [1-2n^{\kappa-\frac{2}{3}}, 1+2n^{\kappa-\frac{2}{3}}]$. Using Proposition 4.7 and the fact that the Airy function is bounded on the real line, we then arrive at Eq. (4.48).

(iv) Finally, (4.18), (4.19), (4.21), (4.22) and Proposition 4.4(iii) lead to,

$$\hat{\phi}_n(x) = \frac{1}{\sqrt{2\pi}}\frac{\varphi(x)^{\frac{1}{2}(\alpha+1)}}{x^{1/4}(x-1)^{1/4}} \exp\left[-\frac{n}{2}\int_1^x \sqrt{\frac{s-1}{s}}h_n(s)\,ds\right](1+\mathcal{O}(n^{-\frac{3}{2}\kappa})),$$
$$= \exp\left[-\frac{n}{2}\int_1^x \sqrt{\frac{s-1}{s}}h_n(s)\,ds\right]\mathcal{O}(x^{\frac{1}{2}\alpha}n^{\frac{1}{6}-\frac{1}{4}\kappa}),$$

as $n \to \infty$, uniformly for $x \in [1+\frac{1}{2}n^{\kappa-\frac{2}{3}}, \infty)$. Since there exists $h_0 > 0$ such that $h_n(s) \geq h_0 > 0$ for n sufficiently large, and as $\frac{1}{\sqrt{s}} \leq \frac{1}{\sqrt{x}}$ for $s \in [1, x]$, one then proves that

$$\exp\left[-\frac{n}{2}\int_1^x \sqrt{\frac{s-1}{s}}h_n(s)\,ds\right] = \mathcal{O}\left(\exp\left[-\frac{h_0}{3}\sqrt{\frac{x-1}{x}}n(x-1)\right]\right)$$
$$= \mathcal{O}(e^{-c(x-1)n^{2/3}}) \qquad (4.53)$$

for some $c > 0$. Inserting this relation into the previous equation it is straightforward to verify that the last part of the Lemma is satisfied, with a different choice of c. □

4.4. Asymptotic Behavior of $\hat{\psi}_r$

The Bessel Region. Here, we will determine the asymptotics of $\hat{\psi}_1$ and $\hat{\psi}_2$ in the Bessel region $(0, n^{-1}]$ using Eq. (4.25). Inserting (4.26) into (4.25), and using the fact that $2^{\alpha\sigma_3}R(0)^{-1}R(x)2^{-\alpha\sigma_3} = I + \mathcal{O}(x/n)$, cf. [Ref. 23, Theorem

3.32], as $n \to \infty$, uniformly for $x \in (0, \delta]$, we obtain

$$\begin{pmatrix} \hat{\psi}_2(x) \\ \hat{\psi}_1(x) \end{pmatrix} = \frac{(-\tilde{f}_n(x))^{1/4} n^{-1/2}}{x(1-x)^{1/4} x^{1/4}} \begin{pmatrix} \frac{\alpha}{4} & \frac{1}{2} \\ -\frac{\alpha}{4} & \frac{1}{2} \end{pmatrix} (\tilde{c}_n n^2)^{-\frac{1}{4}\sigma_3} (I + \mathcal{O}(x/n))$$
$$\times \begin{pmatrix} 1-\alpha & -i(\alpha+1) \\ 1 & i \end{pmatrix} \begin{pmatrix} \sin\zeta_1(x) & \cos\zeta_1(x) \\ -i\sin\zeta_2(x) & -i\cos\zeta_2(x) \end{pmatrix} \begin{pmatrix} J_\alpha(2(-\tilde{f}_n(x))^{1/2}) \\ J'_\alpha(2(-\tilde{f}_n(x))^{1/2}) \end{pmatrix}, \tag{4.54}$$

as $n \to \infty$, uniformly for $x \in (0, \delta]$. Now, since $\sin\zeta_1(x) = 1 + \mathcal{O}(x)$, $\sin\zeta_2(x) = -1 + \mathcal{O}(x)$, $\cos\zeta_1(x) = (\alpha+1)\sqrt{x}(1+\mathcal{O}(x))$, and $\cos\zeta_2(x) = (1-\alpha)\sqrt{x}(1+\mathcal{O}(x))$, as $x \to 0$ (which follows from (4.23)) we have

$$\begin{pmatrix} 1-\alpha & -i(\alpha+1) \\ 1 & i \end{pmatrix} \begin{pmatrix} \sin\zeta_1(x) & \cos\zeta_1(x) \\ -i\sin\zeta_2(x) & -i\cos\zeta_2(x) \end{pmatrix}$$
$$= [2I + \mathcal{O}(x)] \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{x} \end{pmatrix}, \quad \text{as } x \to 0.$$

Inserting this relation into (4.54) and using the fact that

$$\frac{(-\tilde{f}_n(x))^{1/4}}{(1-x)^{1/4} x^{1/4}} = \tilde{c}_n^{1/4} n^{1/2} (1 + \mathcal{O}(x)),$$

we then arrive at

$$\begin{pmatrix} \hat{\psi}_2(x) \\ \hat{\psi}_1(x) \end{pmatrix} = \tilde{c}_n^{1/4} \begin{pmatrix} \frac{\alpha}{2} & 1 \\ -\frac{\alpha}{2} & 1 \end{pmatrix} (\tilde{c}_n n^2)^{-\frac{1}{4}\sigma_3} (I + \mathcal{O}(x)) \begin{pmatrix} \frac{1}{x} J_\alpha(2(-\tilde{f}_n(x))^{1/2}) \\ \frac{1}{\sqrt{x}} J'_\alpha(2(-\tilde{f}_n(x))^{1/2}) \end{pmatrix}, \tag{4.55}$$

as $n \to \infty$, uniformly for $x \in (0, n^{-1}]$.

Now, we split the Bessel region $(0, n^{-1}]$ up into the intervals $(0, n^{-2}]$ and $[n^{-2}, n^{-1}]$, and we determine the asymptotics of $\hat{\psi}_1$ and $\hat{\psi}_2$ in each of these two intervals. From Corollary 4.6 and Eq. (4.55) we have,

$$\begin{pmatrix} \hat{\psi}_2(x) \\ \hat{\psi}_1(x) \end{pmatrix} = \tilde{c}_n^{1/4} \begin{pmatrix} \frac{\alpha}{2} & 1 \\ -\frac{\alpha}{2} & 1 \end{pmatrix} (\tilde{c}_n n^2)^{-\frac{1}{4}\sigma_3} \begin{pmatrix} \frac{1}{x} J_\alpha(2(-\tilde{f}_n(x))^{1/2}) + \mathcal{O}(n^\alpha x^{\frac{\alpha}{2}}) \\ \frac{1}{\sqrt{x}} J'_\alpha(2(-\tilde{f}_n(x))^{1/2}) + \mathcal{O}(n^\alpha x^{\frac{\alpha}{2}}) \end{pmatrix},$$

as $n \to \infty$, uniformly for $x \in (0, n^{-2}]$. Further, from Proposition 4.5 and the fact that $J'_\alpha(z) = -J_{\alpha+1}(z) + \frac{\alpha}{z} J_\alpha(z)$, we then obtain

$$\begin{pmatrix} \hat{\psi}_2(x) \\ \hat{\psi}_1(x) \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} & 1 \\ -\frac{\alpha}{2} & 1 \end{pmatrix} \begin{pmatrix} n^{-1/2} \frac{1}{x} J_\alpha(2\tilde{c}_n^{1/2} n\sqrt{x}) + \mathcal{O}(n^{\alpha-\frac{1}{2}} x^{\frac{\alpha}{2}}) \\ -\frac{\tilde{c}_n^{1/2} n^{1/2}}{\sqrt{x}} J_{\alpha+1}(2\tilde{c}_n^{1/2} n\sqrt{x}) + \frac{\alpha}{2} n^{-1/2} \frac{1}{x} J_\alpha(2\tilde{c}_n^{1/2} n\sqrt{x}) + \mathcal{O}(n^{\alpha+\frac{1}{2}} x^{\frac{\alpha}{2}}) \end{pmatrix},$$

as $n \to \infty$, uniformly for $x \in (0, n^{-2}]$. This gives the asymptotics in the interval $(0, n^{-2}]$. The derivation in the other interval, i.e. $[n^{-2}, n^{-1}]$, is analogous and we obtain the following result.

Lemma 4.9. *As $n \to \infty$,*

$$\hat{\psi}_1(x) = -\frac{\tilde{c}_n^{1/2} n^{1/2}}{\sqrt{x}} J_{\alpha+1}\left(2\tilde{c}_n^{1/2} n\sqrt{x}\right) + \begin{cases} \mathcal{O}(n^{\alpha+\frac{1}{2}} x^{\alpha/2}), & \textit{uniformly for } x \in (0, n^{-2}], \\ \mathcal{O}(x^{-1/4}), & \textit{uniformly for } x \in [n^{-2}, n^{-1}]. \end{cases} \tag{4.56}$$

$$\hat{\psi}_2(x) = \frac{n^{-1/2}\alpha}{x} J_\alpha\left(2\tilde{c}_n^{1/2} n\sqrt{x}\right) - \frac{\tilde{c}_n^{1/2} n^{1/2}}{\sqrt{x}} J_{\alpha+1}\left(2\tilde{c}_n^{1/2} n\sqrt{x}\right) + \begin{cases} \mathcal{O}(n^{\alpha+\frac{1}{2}} x^{\alpha/2}), & \textit{uniformly for } x \in (0, n^{-2}], \\ \mathcal{O}(x^{-1/4}), & \textit{uniformly for } x \in [n^{-2}, n^{-1}]. \end{cases} \tag{4.57}$$

The Airy Region. Inserting (4.28) into (4.25) and using $2^{\alpha\sigma_3} R(0)^{-1} R(x) 2^{-\alpha\sigma_3} = I + \mathcal{O}(1/n)$ we obtain

$$\begin{pmatrix} \hat{\psi}_2(x) \\ \hat{\psi}_1(x) \end{pmatrix} = (-1)^n \frac{n^{-1/2}}{x x^{1/4}} \begin{pmatrix} \frac{\alpha}{4} & \frac{1}{2} \\ -\frac{\alpha}{4} & \frac{1}{2} \end{pmatrix} (\tilde{c}_n n^2)^{-\frac{1}{4}\sigma_3} (I + \mathcal{O}(1/n)) \times \begin{pmatrix} 1-\alpha & -i(\alpha+1) \\ 1 & i \end{pmatrix} \begin{pmatrix} \cos\eta_1(x) & -\frac{\sin\eta_1(x)}{(1-x)^{1/2}} \\ -i\cos\eta_2(x) & i\frac{\sin\eta_2(x)}{(1-x)^{1/2}} \end{pmatrix} \left| \frac{f_n(x)}{x-1} \right|^{\frac{1}{4}\sigma_3} \times \begin{pmatrix} \mathrm{Ai}\,(f_n(x)) \\ \mathrm{Ai}'(f_n(x)) \end{pmatrix}, \tag{4.58}$$

as $n \to \infty$, uniformly for $x \in [1-\delta, 1+\delta]$. From Eq. (4.20) and Proposition 4.7 we then obtain

$$\begin{aligned}
\begin{pmatrix} \hat{\psi}_2(x) \\ \hat{\psi}_1(x) \end{pmatrix} &= (-1)^n n^{-1/2} \begin{pmatrix} \frac{\alpha}{4} & \frac{1}{2} \\ -\frac{\alpha}{4} & \frac{1}{2} \end{pmatrix} (\tilde{c}_n n^2)^{-\frac{1}{4}\sigma_3} (I + \mathcal{O}(n^{\kappa - \frac{2}{3}})) \\
&\quad \times \begin{pmatrix} 1-\alpha & -i(\alpha+1) \\ 1 & i \end{pmatrix} \begin{pmatrix} 1 & -(\alpha+1) \\ -i & i(\alpha-1) \end{pmatrix} \left| \frac{f_n(x)}{x-1} \right|^{\frac{1}{4}\sigma_3} \\
&\quad \times \begin{pmatrix} \mathrm{Ai}\,(f_n(x)) \\ \mathrm{Ai}'(f_n(x)) \end{pmatrix} \\
&= (-1)^n n^{-1/2} \begin{pmatrix} \frac{\alpha}{2} & 1 \\ -\frac{\alpha}{2} & 1 \end{pmatrix} (\tilde{c}_n n^2)^{-\frac{1}{4}\sigma_3} (I + \mathcal{O}(n^{\kappa - \frac{2}{3}})) \\
&\quad \times \begin{pmatrix} -\alpha & (\alpha^2 - 1) \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} c_n^{1/4} n^{1/6} \mathrm{Ai}\,(c_n n^{2/3}(x-1)) + \mathcal{O}(n^{-1/2+\frac{9}{4}\kappa}) \\ \mathcal{O}(n^{-1/6+\frac{1}{4}\kappa}) \end{pmatrix},
\end{aligned}$$

as $n \to \infty$, uniformly for $x \in [1 - n^{\kappa - \frac{2}{3}}, 1 + n^{\kappa - \frac{2}{3}}]$, which implies after a straightforward calculation,

$$\begin{pmatrix} \hat{\psi}_2(x) \\ \hat{\psi}_1(x) \end{pmatrix} = (-1)^n \begin{pmatrix} \frac{\alpha}{2} & 1 \\ -\frac{\alpha}{2} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}(n^{-5/6}) \\ (c_n \tilde{c}_n)^{1/4} n^{1/6} \mathrm{Ai}\,(c_n n^{2/3}(x-1)) + \mathcal{O}(n^{-1/6+\frac{1}{4}\kappa}) \end{pmatrix}.$$

We now have proved the following Lemma.

Lemma 4.10. *Let $r = 1$ or 2. As $n \to \infty$,*

$$\hat{\psi}_r(x) = (-1)^n (c_n \tilde{c}_n)^{1/4} n^{1/6} \mathrm{Ai}\,(c_n n^{2/3}(x-1)) + \mathcal{O}(n^{-1/6+\frac{1}{4}\kappa}), \qquad (4.59)$$

uniformly for $x \in [1 - n^{\kappa - \frac{2}{3}}, 1 + n^{\kappa - \frac{2}{3}}]$.

The Bulk Region. From Eqs. (4.54) and (4.30), (4.58) and (4.31), (4.25) and (4.27), we obtain

$$\begin{pmatrix}\hat\psi_2(x)\\ \hat\psi_1(x)\end{pmatrix} = \frac{(-1)^n n^{-1/2}}{\sqrt{\pi x}(1-x)^{1/4}x^{1/4}}\begin{pmatrix}\frac{\alpha}{4} & \frac12\\ -\frac{\alpha}{4} & \frac12\end{pmatrix}(\tilde c_n n^2)^{-\frac14\sigma_3}(I+\mathcal{O}(x/n))$$
$$\times\begin{pmatrix}1-\alpha & -i(\alpha+1)\\ 1 & i\end{pmatrix}$$
$$\times\begin{cases}\begin{pmatrix}\cos F_{n,1}(x)+\tau_n(x)\sin F_{n,1}(x)+\mathcal{O}(1/n)\\ -i(\cos F_{n,2}(x)+\tau_n(x)\sin F_{n,2}(x))+\mathcal{O}(1/n)\end{pmatrix}, & \text{uniformly for } x\in[\frac12 n^{-1},\delta],\\ \begin{pmatrix}\cos F_{n,1}(x)+\mathcal{O}\left(\frac{1}{n(1-x)^{3/2}}\right)\\ -i\cos F_{n,2}(x)+\mathcal{O}\left(\frac{1}{n(1-x)^{3/2}}\right)\end{pmatrix}, & \text{uniformly for } x\in[\delta, 1-\frac12 n^{\kappa-\frac23}].\end{cases}$$

By (4.15)

$$\cos\left(\frac12 F_{n,1}(x)-\frac12 F_{n,2}(x)\right) = \cos\left(\frac12\eta_1(x)-\frac12\eta_2(x)\right)$$
$$= \cos\left(\frac12\arccos(2x-1)\right) = \sqrt{x},$$

and hence

$$\cos F_{n,1}(x)+\cos F_{n,2}(x) = 2\sqrt{x}\cos G_n(x),$$
$$\sin F_{n,1}(x)+\sin F_{n,2}(x) = 2\sqrt{x}\sin G_n(x),$$

with $G_n(x)=\frac12 F_{n,1}(x)+\frac12 F_{n,2}(x)$. Using the fact that $\tau_n(x)=\mathcal{O}(\frac{1}{n\sqrt{x}}$ uniformly for $x\in[\frac12 n^{-1},\delta]$, as $n\to\infty$, we obtain

$$\begin{pmatrix}\hat\psi_2(x)\\ \hat\psi_1(x)\end{pmatrix} = \frac{(-1)^n n^{-1/2}}{\sqrt{\pi x}(1-x)^{1/4}x^{1/4}}\begin{pmatrix}\frac{\alpha}{2} & 1\\ -\frac{\alpha}{2} & 1\end{pmatrix}(\tilde c_n n^2)^{-\frac14\sigma_3}$$
$$\times\begin{pmatrix}\mathcal{O}(1)\\ \sqrt{x}\cos G_n(x)+\mathcal{O}(\frac{1}{n(1-x)^{3/2}})\end{pmatrix},$$

uniformly for $x\in[\frac12 n^{-1}, 1-\frac12 n^{\kappa-\frac23}]$, as $n\to\infty$. We then arrive at the following result.

Lemma 4.11. *Let $r=1$ or 2. As $n\to\infty$, uniformly for $x\in[\frac12 n^{-1}, 1-\frac12 n^{\kappa-\frac23}]$,*

$$\hat\psi_r(x) = \frac{(-1)^n\tilde c_n^{1/4}}{\sqrt{\pi}x^{3/4}(1-x)^{1/4}}\cos G_n(x)+\mathcal{O}\left(\frac{1}{nx^{5/4}(1-x)^{7/4}}\right), \quad (4.60)$$

with

$$G_n(x) = -\frac{n}{2}\int_1^x \sqrt{\frac{1-s}{s}}\, h_n(s)\, ds + \frac{1}{2}\alpha \arccos(2x-1) - \frac{\pi}{4}. \qquad (4.61)$$

The Exponential Region. As in the proof of Lemma 4.8(iv) we obtain from (4.25), (4.28), (4.29), Proposition 4.4(iii) and (4.53), the following result.

Lemma 4.12. *Let $r = 1$ or 2. There exists a constant $c > 0$ such that*

$$\hat{\psi}_r(x) = \mathcal{O}(e^{-c(x-1)n^{2/3}}), \qquad (4.62)$$

as $n \to \infty$, uniformly for $x \in [1 + n^{\kappa - \frac{2}{3}}, \infty)$.

5. ASYMPTOTICS OF THE MATRIX B

We determine the asymptotics of the matrix B by following and occasionally streamlining the path first developed in [Ref. 7, Sec. 4.2].

The following representations of the entries of B are straightforward to verify.

$$\langle \varepsilon\phi_q, \phi_p\rangle = \sqrt{\beta_p\beta_q}\left[\frac{1}{2}\int_0^\infty \hat{\phi}_p(x)\, dx \int_0^\infty \hat{\phi}_q(x)\, dx - \int_0^\infty \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_q}}^\infty \hat{\phi}_q(y)\, dy\, dx\right], \qquad (5.1)$$

$$\langle \varepsilon\psi_r, \phi_p\rangle = \sqrt{\beta_p\beta_n}\left[\frac{1}{2}\int_0^\infty \hat{\phi}_p(x)\, dx \int_0^\infty \hat{\psi}_r(x)\, dx - \int_0^\infty \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_n}}^\infty \hat{\psi}_r(y)\, dy\, dx\right], \qquad (5.2)$$

$$\langle \varepsilon\psi_1, \psi_2\rangle = \beta_n\left[\frac{1}{2}\int_0^\infty \hat{\psi}_1(x)\, dx \int_0^\infty \hat{\psi}_2(x)\, dx - \int_0^\infty \hat{\psi}_2(x) \int_x^\infty \hat{\psi}_1(y)\, dy\, dx\right], \qquad (5.3)$$

with $p, q \in \mathbb{N}$ and $r \in \{1, 2\}$, and where $\hat{\phi}_n$ and $\hat{\psi}_r$ are defined in (4.1). Thus, in order to obtain the asymptotic behavior of the matrix B we need to determine the asymptotic behavior of the single and double integrals appearing in these three equations which will be done in Sec. 5.1 and 5.2 respectively. As noted at the beginning of Sec. 4 we do this by splitting $(0, \infty)$ into four regions $(0, n^{-1}]$, $[n^{-1}, 1 - n^{\kappa-\frac{2}{3}}]$, $[1 - n^{\kappa-\frac{2}{3}}, 1 + n^{\kappa-\frac{2}{3}}]$ and $[1 + n^{\kappa-\frac{2}{3}}, \infty)$, with $\kappa = \frac{1}{12}$ fixed, and integrate separately over each of these four regions. In the final and brief Sec. 5.3

we summarize our results in such a way that the asymptotic result for the matrix B stated in Lemma 2.6 is apparent.

5.1. The Single Integrals

We start with the following three auxiliary Propositions, which will also be used to determine the asymptotic behavior of the double integrals.

Proposition 5.1. *The first and second derivatives of F_n and G_n, defined in* (4.15) *and* (4.61)*, satisfy,*

$$\frac{1}{Z_n'(x)} = -\frac{2}{h_n(x)} \frac{x^{1/2}}{n(1-x)^{1/2}} \left[1 + \mathcal{O}\left(\frac{1}{n(1-x)}\right)\right], \tag{5.4}$$

$$Z_n''(x) = \mathcal{O}\left(\frac{n}{x^{3/2}(1-x)^{1/2}}\right)\left[1 + \mathcal{O}\left(\frac{1}{n(1-x)}\right)\right], \tag{5.5}$$

as $n \to \infty$, uniformly for $x \in (0, 1)$, where $Z \in \{F, G\}$.

Proof: We will prove the result for F_n. The result for G_n then also follows since G_n equals F_n with α replaced by $\alpha - 1$. The first derivative of F_n can be explicitly determined from the definition (4.15),

$$\frac{1}{F_n'(x)} = -\frac{2}{h_n(x)} \frac{x^{1/2}}{n(1-x)^{1/2}} \left(1 - \frac{\alpha+1}{nh_n(x)(1-x)+\alpha+1}\right).$$

Since $h_n(x) \geq h_0 > 0$ for n sufficiently large, $x \in [0, \infty)$, see Sec. 4.1 under (4.7), we have $|nh_n(x)(1-x) + \alpha + 1| \geq nh_0(1-x)$ for all n sufficiently large, $x \in (0, 1)$, which proves (5.4). Similarly, it follows from

$$F_n''(x) = \frac{n}{x^{3/2}(1-x)^{1/2}} \left(-\frac{1}{2}h_n'(x)x(1-x) + \frac{1}{4}h_n(x) + \frac{1}{4}(\alpha+1)\frac{1-2x}{n(1-x)}\right)$$

that (5.5) is satisfied as well. □

Proposition 5.2. *As $n \to \infty$, uniformly for $x \in [n^{-1}, 1 - \frac{1}{2}n^{\kappa - \frac{2}{3}}]$,*

$$\frac{1}{F_n'(x)x^{1/4}(1-x)^{1/4}} = \mathcal{O}(n^{-1/2-\frac{3}{4}\kappa}), \tag{5.6}$$

$$\frac{1}{G_n'(x)x^{3/4}(1-x)^{1/4}} = \mathcal{O}(n^{-1/2-\frac{3}{4}\kappa}), \tag{5.7}$$

$$\left(\frac{1}{F_n'(x)x^{1/4}(1-x)^{1/4}}\right)' = \mathcal{O}\left(\frac{1}{nx^{3/4}(1-x)^{7/4}}\right), \tag{5.8}$$

$$\left(\frac{1}{G_n'(x)x^{3/4}(1-x)^{1/4}}\right)' = \mathcal{O}\left(\frac{1}{nx^{5/4}(1-x)^{7/4}}\right). \tag{5.9}$$

Proof: Equations (5.6) and (5.7) follow from (5.4) and from the fact that $h_n(x) \geq h_0 > 0$ for n sufficiently large, $x \in [0, \infty)$. Further, since

$$\left(\frac{1}{F_n'(x)x^{1/4}(1-x)^{1/4}}\right)'$$
$$= -\frac{1}{F_n'(x)^2}F_n''(x)x^{-1/4}(1-x)^{-1/4} - \frac{1}{4}\frac{1}{F_n'(x)}x^{-5/4}(1-x)^{-5/4}(1-2x),$$

Equation (5.8) follows from Eqs. (5.4) and (5.5). The proof of the last equation of the Proposition is similar. □

Proposition 5.3. *As $n \to \infty$, uniformly for $a, b \in [n^{-1}, 1 - \frac{1}{2}n^{\kappa - \frac{2}{3}}]$,*

$$\int_a^b \frac{\cos F_n(y)}{y^{1/4}(1-y)^{1/4}}\,dy = \mathcal{O}(n^{-1/2-\frac{3}{4}\kappa}), \tag{5.10}$$

$$\int_a^b \frac{\cos G_n(y)}{y^{3/4}(1-y)^{1/4}}\,dy = \mathcal{O}(n^{-1/2-\frac{3}{4}\kappa}). \tag{5.11}$$

Proof: This is immediate after integrating by parts and using Proposition 5.2. □

We now have the necessary ingredients to determine the asymptotic behavior of the single integrals.

5.1.1. Integrals Involving $\hat{\phi}_n$

Proposition 5.4. *As $n \to \infty$,*

(i) *Bessel, bulk and exponential region: there exists a constant* $c > 0$ *such that,*

$$\int_0^x \left|\hat{\phi}_n(y)\right| dy = \mathcal{O}(n^{-3/4}), \quad \textit{uniformly for } x \in (0, n^{-1}], \tag{5.12}$$

$$\int_{n^{-1}}^x \hat{\phi}_n(y)\, dy = \mathcal{O}(n^{-1/2-\frac{3}{4}\kappa}), \quad \textit{uniformly for } x \in [n^{-1}, 1 - n^{\kappa-\frac{2}{3}}], \tag{5.13}$$

$$\int_x^\infty \left|\hat{\phi}_n(y)\right| dy = \mathcal{O}(e^{-cn^\kappa}), \quad \textit{uniformly for } x \in [1 + n^{\kappa-\frac{2}{3}}, \infty). \tag{5.14}$$

(ii) *Airy region:*

$$\int_{1-n^{\kappa-\frac{2}{3}}}^x \hat{\phi}_n(y)\, dy = \mathcal{O}(n^{-1/2}), \quad \textit{uniformly for } x \in [1 - n^{\kappa-\frac{2}{3}}, 1+n^{\kappa-\frac{2}{3}}], \tag{5.15}$$

$$\int_{1-n^{\kappa-\frac{2}{3}}}^{1+n^{\kappa-\frac{2}{3}}} \hat{\phi}_n(y)\, dy = \sqrt{2} c_n^{-3/4} n^{-1/2} + \mathcal{O}(n^{-1/2-\frac{3}{4}\kappa}). \tag{5.16}$$

Proof: (i) Equation (5.12) is immediate from (4.46), Eq. (5.13) follows from (4.47) and (5.10), and Eq. (5.14) follows from (4.49).

(ii) From the asymptotic behavior (4.48) of $\hat{\phi}_n$ in the Airy region we obtain,

$$\int_{1-n^{\kappa-\frac{2}{3}}}^x \hat{\phi}_n(y)\, dy = \sqrt{2} c_n^{-3/4} n^{-1/2} \int_{-c_n n^\kappa}^{c_n n^{\frac{2}{3}}(x-1)} \mathrm{Ai}\,(u)\, du + \mathcal{O}(n^{-5/6+\frac{5}{4}\kappa}), \tag{5.17}$$

as $n \to \infty$, uniformly for $x \in [1 - n^{\kappa-\frac{2}{3}}, 1 + n^{\kappa-\frac{2}{3}}]$. Since $\int_a^b \mathrm{Ai}\,(u)\, du$ is uniformly bounded for $a, b \in \mathbb{R}$, see e.g. [Ref. 1, (10.4.82) and (10.4.83)], this yields (5.15). Next, note that $\int_{-\infty}^\infty \mathrm{Ai}\,(t)\, dt = 1$, $\int_{-\infty}^{-y} \mathrm{Ai}\,(t)\, dt = \mathcal{O}(y^{-3/4})$ and $\int_y^\infty \mathrm{Ai}\,(t)\, dt = \mathcal{O}(e^{-cy})$ as $y \to \infty$ for some $c > 0$, see again [Ref. 1, (10.4.82) and (10.4.83)], implying

$$\int_{-c_n n^\kappa}^{c_n n^\kappa} \mathrm{Ai}\,(u)\, du = 1 + \mathcal{O}(n^{-\frac{3}{4}\kappa}).$$

Together with (5.17) this proves the remaining statement (5.16) of the Proposition. □

Lemma 5.5. *There exists* $0 < \tau = \tau(m, \alpha) < 1$ *such that*

$$\int_0^{\infty} \hat{\phi}_n(y)\, dy = \left(\frac{1}{\sqrt{m}} + \mathcal{O}(n^{-\tau}) \right) n^{-1/2}, \quad \text{as } n \to \infty, \tag{5.18}$$

$$\int_a^{b} \hat{\phi}_n(y)\, dy = \mathcal{O}(n^{-1/2}), \quad \text{as } n \to \infty, \text{ uniformly for } a, b \in [0, \infty]. \tag{5.19}$$

Proof: The Lemma is immediate from the previous Proposition together with the fact that $c_n = (2m)^{2/3} + \mathcal{O}(n^{-1/m})$ as $n \to \infty$, see (4.11). □

5.1.2. Integrals Involving $\hat{\psi}_r$

Proposition 5.6. *Let* $r \in \{1, 2\}$*. As* $n \to \infty$*,*

(i) *Bulk and exponential region: there exists a constant* $c > 0$ *such that,*

$$\int_{n^{-1}}^{x} \hat{\psi}_r(y)\, dy = \mathcal{O}(n^{-1/2-\frac{3}{4}\kappa}), \quad \text{uniformly for } x \in [n^{-1}, 1 - n^{\kappa-\frac{2}{3}}], \tag{5.20}$$

$$\int_x^{\infty} \left| \hat{\psi}_r(y) \right| dy = \mathcal{O}(e^{-cn^{\kappa}}), \quad \text{uniformly for } x \in [1 + n^{\kappa-\frac{2}{3}}, \infty]. \tag{5.21}$$

(ii) *Bessel region:*

$$\int_0^{x} \hat{\psi}_r(y) dy = \mathcal{O}(n^{-1/2}), \quad \text{uniformly for } x \in (0, n^{-1}], \tag{5.22}$$

$$\int_0^{n^{-1}} \hat{\psi}_r(y)\, dy = (-1)^r n^{-1/2} + \mathcal{O}(n^{-3/4}). \tag{5.23}$$

(iii) *Airy region:*

$$\int_{1-n^{\kappa-\frac{2}{3}}}^{x} \hat{\psi}_r(y)\, dy = \mathcal{O}(n^{-1/2}), \quad \text{uniformly for } x \in [1 - n^{\kappa-\frac{2}{3}}, 1 + n^{\kappa-\frac{2}{3}}], \tag{5.24}$$

$$\int_{1-n^{\kappa-\frac{2}{3}}}^{1+n^{\kappa-\frac{2}{3}}} \hat{\psi}_r(y)\, dy = (-1)^n \tilde{c}_n^{1/4} c_n^{-3/4} n^{-1/2} + \mathcal{O}(n^{-1/2-\frac{3}{4}\kappa}). \tag{5.25}$$

Proof: (i) Equation (5.20) is immediate from (4.60) and (5.11), and Eq. (5.21) follows from (4.62).

(ii) From the asymptotic behavior (4.56) of $\hat{\psi}_1$ in the Bessel region we obtain,

$$\int_0^x \hat{\psi}_1(y)\,dy = -n^{-1/2}\int_0^{2\tilde{c}_n^{1/2}n\sqrt{x}} J_{\alpha+1}(t)\,dt + \mathcal{O}(n^{-3/4}), \tag{5.26}$$

as $n \to \infty$, uniformly for $x \in (0, n^{-1}]$. From [Ref. 1, (9.2.1) and (11.4.17)] we learn that $\int_0^\infty J_{\alpha+1}(t)\,dt = 1$ and $\int_y^\infty J_{\alpha+1}(t)\,dt = \mathcal{O}(y^{-1/2})$ as $y \to \infty$. Together with (5.26) this yields (5.22) as well as (5.23) for the case $r = 1$. The case $r = 2$ can be proven similarly using the asymptotic behavior (4.57) of $\hat{\psi}_2$ in the Bessel region together with the previous facts about Bessel integrals as well as the fact $\int_0^\infty t^{-1}J_\alpha(t)\,dt = 1/\alpha$, see [Ref. 1, (11.4.16)].

(iii) Finally, the proof of the last part of the Proposition is analogous to the proof of Proposition 5.4(ii) using the asymptotic behavior (4.59) of $\hat{\psi}_r$ in the Airy region. □

Lemma 5.7. *Let* $r \in \{1, 2\}$. *There exists* $0 < \tau = \tau(m, \alpha) < 1$ *such that,*

$$\int_0^\infty \hat{\psi}_r(x)\,dx = \left((-1)^r + \frac{(-1)^n}{\sqrt{2m-1}} + \mathcal{O}(n^{-\tau})\right) n^{-1/2}, \quad \text{as } n \to \infty, \tag{5.27}$$

$$\int_a^b \hat{\psi}_r(x)\,dx = \mathcal{O}(n^{-1/2}), \quad \text{as } n \to \infty \text{ uniformly for } a, b \in [0, \infty]. \tag{5.28}$$

Proof: The Lemma is immediate from the previous Proposition together with the facts that $c_n = (2m)^{2/3} + \mathcal{O}(n^{-1/m})$ and $\tilde{c}_n = \left(\frac{2m}{2m-1}\right)^2 + \mathcal{O}(n^{-1/m})$ as $n \to \infty$, see (4.11) and (4.12). □

5.2. The Double Integrals

The goal of this subsection is to determine the asymptotic behaviour of the double integrals appearing in (5.1)–(5.3). Following[7] we decompose the range of integration $\mathbb{R}_+$ of the outer integral into two regions, namely into the bulk region which is essentially given by $(n^{-1}, 1 - n^{\kappa - \frac{2}{3}})$ and its complement. We first determine the contribution from the region outside the bulk in Sec. 5.2.1. As in Ref. 7 a more subtle argument is needed to determine the leading order asymptotics in the oscillatory bulk region in Sec. 5.2.2. An important ingredient in the argument is Proposition 5.13 which provides a surprisingly simple description of the phase deviations of orthogonal polynomials with different degrees in the oscillatory region. Such a formula was first presented in [Ref. 7, Lemma 4.7]. The formula follows from a special property of the equilibrium measure stated in Proposition 5.12 (see [Ref. 7, Lemma 4.8] for the corresponding property in the Hermite case). Our results on the double integrals are summarized in Sec. 5.2.3.

5.2.1. The Double Integrals Outside the Bulk

We start with the following technical Propositions.

Proposition 5.8. *Let $p = n + i$ and $q = n + j$ with i, j some fixed integers. Then,*

$$\frac{\beta_p}{\beta_q} = 1 + \frac{1}{m}\frac{p-q}{q} + \mathcal{O}(n^{-1-1/m}), \quad \text{as } n \to \infty. \tag{5.29}$$

In particular, $\frac{\beta_p}{\beta_q} = 1 + \mathcal{O}(1/n)$ as $n \to \infty$.

Proof: The proof is similar to the proof of [Ref. 7, Lemma 4.4]. Recall from (4.4) that $\beta_n = \sum_{k=-1}^{\infty} \beta_{(k)} n^{-k/m}$. Since $p^{-c} - q^{-c} = \mathcal{O}(n^{-1-c})$ as $n \to \infty$ (for $c > 0$) we then obtain,

$$\begin{aligned}\beta_p - \beta_q &= \beta_{(-1)}(p^{1/m} - q^{1/m}) + \sum_{k=1}^{m} \beta_{(k)}(p^{-k/m} - q^{-k/m}) + \mathcal{O}(n^{-1-1/m}) \\ &= \beta_{(-1)}(p^{1/m} - q^{1/m}) + \mathcal{O}(n^{-1-1/m}).\end{aligned}$$

This implies,

$$\begin{aligned}\frac{\beta_p}{\beta_q} - 1 = \frac{\beta_p - \beta_q}{\beta_q} &= \left[\frac{p^{1/m} - q^{1/m}}{q^{1/m}} + \mathcal{O}(n^{-1-\frac{2}{m}})\right]\left(1 + \mathcal{O}(n^{-1/m})\right) \\ &= \left[\left(1 + \frac{p-q}{q}\right)^{1/m} - 1 + \mathcal{O}(n^{-1-\frac{2}{m}})\right](1 + \mathcal{O}(n^{-1/m})).\end{aligned}$$

The Proposition now follows by expanding the $1/m$-th power at 1. □

Proposition 5.9. *Let $p = n + i$ and $q = n + j$ with i, j some fixed integers and define for $u \in \mathbb{R}$,*

$$u_{p,q} = c_q q^{2/3}\left(\frac{\beta_p}{\beta_q} - 1\right) + \frac{c_q q^{2/3}}{c_p p^{2/3}}\frac{\beta_p}{\beta_q} u. \tag{5.30}$$

Then,

$$2\int_{-c_p p^\kappa}^{c_p p^\kappa} \mathrm{Ai}\,(u) \int_{u_{p,q}}^{c_q q^\kappa} \mathrm{Ai}\,(v)\, dv\, du = 1 + \mathcal{O}(n^{-\frac{3}{4}\kappa}), \quad \text{as } n \to \infty. \tag{5.31}$$

Proof: As in the previous Proposition one can verify that $\frac{c_q}{c_p} = 1 + \mathcal{O}(n^{-1-1/m})$ as $n \to \infty$. By (5.29) we then obtain $u_{p,q} = u + \mathcal{O}(n^{-1/3}) + \mathcal{O}(un^{-1})$ as $n \to \infty$.

Together with the boundedness (on the real line) of the Airy function, this yields

$$\int_{u_{p,q}}^{c_q q^\kappa} \mathrm{Ai}\,(v)\,dv = \int_u^{c_q q^\kappa} \mathrm{Ai}\,(v)\,dv + \mathcal{O}(n^{-1/3}) + \mathcal{O}(un^{-1}).$$

Then, since $|\mathrm{Ai}\,(t)| \le C(1+|t|)^{-1/4}$ and $|t\mathrm{Ai}\,(t)| \le C|t|^{3/4}$ for $t \in \mathbb{R}$ and $C > 0$ some constant, we obtain,

$$\begin{aligned} &2\int_{-c_p p^\kappa}^{c_p p^\kappa} \mathrm{Ai}\,(u) \int_{u_{p,q}}^{c_q q^\kappa} \mathrm{Ai}\,(v)\,dv\,du \\ &= 2\int_{-c_p p^\kappa}^{c_p p^\kappa} \mathrm{Ai}\,(u) \int_u^{c_q q^\kappa} \mathrm{Ai}\,(v)\,dv\,du + \mathcal{O}(n^{-\frac{1}{3}+\frac{3}{4}\kappa}) \\ &= \left(\int_{-c_p p^\kappa}^{c_q q^\kappa} \mathrm{Ai}\,(v)\,dv\right)^2 - \left(\int_{c_p p^\kappa}^{c_q q^\kappa} \mathrm{Ai}\,(v)\,dv\right)^2 + \mathcal{O}(n^{-\frac{1}{3}+\frac{3}{4}\kappa}). \end{aligned}$$

Since the Airy function is bounded on the real line we have

$$\int_{c_p p^\kappa}^{c_q q^\kappa} \mathrm{Ai}\,(v)\,dv = \mathcal{O}(n^{\kappa-1}).$$

As in the proof of Proposition 5.4(ii) we obtain

$$\int_{-c_p p^\kappa}^{c_q q^\kappa} \mathrm{Ai}\,(v)\,dv = 1 + \mathcal{O}(n^{-\frac{3}{4}\kappa}).$$

This proves the Proposition. □

Proposition 5.10. *As $n \to \infty$,*

$$\int_0^{2\tilde{c}_n^{1/2}\sqrt{n}} J'_\alpha(u) \int_u^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(v)\,dv\,du = \frac{1}{2} + \mathcal{O}(n^{-1/2}), \tag{5.32}$$

$$\int_0^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(u) \int_u^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(v)\,dv\,du = \frac{1}{2} + \mathcal{O}(n^{-1/4}). \tag{5.33}$$

Proof: Integrating by parts and using $J_\alpha(0) = 0$ for $\alpha > 0$ and $\int_0^\infty J_\alpha(u)J_{\alpha+1}(u)du = 1/2$, see e.g. [Ref. 1, (11.4.42)], we obtain

$$\begin{aligned} \int_0^{2\tilde{c}_n^{1/2}\sqrt{n}} J'_\alpha(u) \int_u^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(v)\,dv\,du &= \int_0^{2\tilde{c}_n^{1/2}\sqrt{n}} J_\alpha(u)J_{\alpha+1}(u)du \\ &= \frac{1}{2} - \int_{2\tilde{c}_n^{1/2}\sqrt{n}}^\infty J_\alpha(u)J_{\alpha+1}(u)du. \end{aligned} \tag{5.34}$$

From [Ref. 1, (9.2.1)] we have $J_\alpha(u)J_{\alpha+1}(u) = -\frac{\cos(2u-\alpha\pi)}{\pi u} + \mathcal{O}(u^{-2})$ as $u \to \infty$. Integrating by parts one can verify that

$$\int_{2\tilde{c}_n^{1/2}\sqrt{n}}^{\infty} \frac{\cos(2u-\alpha\pi)}{\pi u} du = \mathcal{O}(n^{-1/2}), \quad \text{as } n \to \infty,$$

so that also

$$\int_{2\tilde{c}_n^{1/2}\sqrt{n}}^{\infty} J_\alpha(u)J_{\alpha+1}(u) du = \mathcal{O}(n^{-1/2}), \quad \text{as } n \to \infty.$$

Inserting these estimates into (5.34) the proof of the first part of the Proposition follows. Next,

$$\int_0^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(u) \int_u^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(v)\, dv\, du = \frac{1}{2}\int_0^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(v)\, dv.$$

Since $\int_0^\infty J_{\alpha+1}(u)du = 1$ and $\int_x^\infty J_{\alpha+1}(u)du = \mathcal{O}(x^{-1/2})$ as $x \to \infty$, see Proof of Proposition 5.6(ii), this yields (5.33), and the Proposition is proven. □

Now, we have the necessary ingredients to determine the asymptotic behavior of the double integrals in (5.1)–(5.3), except for the part of the outer integral which lies in the bulk.

Proposition 5.11. *Let $p = n + i$ and $q = n + j$ with i, j some fixed integers and let $r \in \{1, 2\}$. There exists $0 < \tau = \tau(m, \alpha) < 1$ such that as $n \to \infty$,*

$$\int_0^\infty \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_q}}^\infty \hat{\phi}_q(y)\, dydx$$
$$= \frac{1}{2m}n^{-1} + \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_q}}^\infty \hat{\phi}_q(y)dydx + \mathcal{O}(n^{-1-\tau}), \tag{5.35}$$

$$\int_0^\infty \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_n}}^\infty \hat{\psi}_r(y)dydx$$
$$= (-1)^n \frac{1}{2m}\sqrt{\frac{m}{2m-1}}n^{-1} + \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_n}}^\infty \hat{\psi}_r(y)dydx + \mathcal{O}(n^{-1-\tau}), \tag{5.36}$$

and

$$\int_0^\infty \hat{\psi}_2(x)\int_x^\infty \hat{\psi}_1(y)dydx = \left(-\frac{3}{2}+\frac{(-1)^n}{\sqrt{2m-1}}+\frac{1}{2}\frac{1}{2m-1}\right)n^{-1} + \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \hat{\psi}_2(x)\int_x^\infty \hat{\psi}_1(y)\,dydx + \mathcal{O}(n^{-1-\tau}). \tag{5.37}$$

Proof: From (5.12), (5.14) and (5.19) one concludes,

$$\int_0^\infty \hat{\phi}_p(x)\int_{x\frac{\beta_p}{\beta_q}}^\infty \hat{\phi}_q(y)dydx = \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \hat{\phi}_p(x)\int_{x\frac{\beta_p}{\beta_q}}^\infty \hat{\phi}_q(y)dydx + \int_{1-p^{\kappa-\frac{2}{3}}}^{1+p^{\kappa-\frac{2}{3}}} \hat{\phi}_p(x)\int_{x\frac{\beta_p}{\beta_q}}^\infty \hat{\phi}_q(y)dydx + \mathcal{O}(n^{-1-\frac{1}{4}}). \tag{5.38}$$

For notational convenience we denote the second double integral on the right hand side of (5.38) by J. From Eqs. (5.14) and (5.19), and from the asymptotic behavior (4.48) of $\hat{\phi}_p$ in the Airy region, we have

$$J = \int_{1-p^{\kappa-\frac{2}{3}}}^{1+p^{\kappa-\frac{2}{3}}} \hat{\phi}_p(x)\int_{x\frac{\beta_p}{\beta_q}}^{1+q^{\kappa-\frac{2}{3}}} \hat{\phi}_q(y)dydx + \mathcal{O}(e^{-cn^\kappa})$$

$$= \sqrt{2}c_p^{1/4}p^{\frac{1}{6}}\int_{1-p^{\kappa-\frac{2}{3}}}^{1+p^{\kappa-\frac{2}{3}}} \mathrm{Ai}\,(c_p p^{\frac{2}{3}}(x-1))\int_{x\frac{\beta_p}{\beta_q}}^{1+q^{\kappa-\frac{2}{3}}} \hat{\phi}_q(y)\,dydx + \mathcal{O}(n^{-4/3+\frac{5}{4}\kappa}).$$

Using Proposition 5.8 one can verify that $x\frac{\beta_p}{\beta_q} \in [1-2q^{\kappa-\frac{2}{3}}, 1+2q^{\kappa-\frac{2}{3}}]$ for n large enough, so that, from (4.48) and from the fact that the Airy function is bounded on the real line,

$$J = 2(c_p c_q)^{\frac{1}{4}} (pq)^{\frac{1}{6}} \int_{1-p^{\kappa-\frac{2}{3}}}^{1+p^{\kappa-\frac{2}{3}}} \mathrm{Ai}\,(c_p p^{\frac{2}{3}}(x-1))$$

$$\times \int_{x\frac{\beta_p}{\beta_q}}^{1+q^{\kappa-\frac{2}{3}}} \mathrm{Ai}\,(c_q q^{\frac{2}{3}}(y-1))\,dydx + \mathcal{O}(n^{-\frac{4}{3}+\frac{9}{4}\kappa})$$

$$= \frac{(c_p c_q)^{-3/4}}{(pq)^{1/2}} \left(2\int_{-c_p p^{\kappa}}^{c_p p^{\kappa}} \mathrm{Ai}\,(u) \int_{u_{p,q}}^{c_q q^{\kappa}} \mathrm{Ai}\,(v)\,dvdu \right) + \mathcal{O}(n^{-\frac{4}{3}+\frac{9}{4}\kappa}), \qquad (5.39)$$

with $u_{p,q}$ defined by (5.30). Proposition 5.9 and (4.11) yield (5.35). The proof of (5.36) is analogous.

It now remains to prove (5.37). Note that as in the proof of (5.35) and (5.36), the reader can verify that

$$\int_{1-n^{\kappa-\frac{2}{3}}}^{1+n^{\kappa-\frac{2}{3}}} \hat{\psi}_2(x) \int_x^{\infty} \hat{\psi}_1(y)dydx = \frac{1}{2}\tilde{c}_n^{1/2} c_n^{-3/2} n^{-1} + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}).$$

Further, from Proposition 5.6 one has,

$$\int_0^{n^{-1}} \hat{\psi}_2(x)\,dx \int_{n^{-1}}^{\infty} \hat{\psi}_1(y)dy = (-1)^n \tilde{c}_n^{1/4} c_n^{-3/4} n^{-1} + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}).$$

The previous two equations together with (5.21) yield

$$\int_0^{\infty} \hat{\psi}_2(x) \int_x^{\infty} \hat{\psi}_1(y)dydx = \left((-1)^n \tilde{c}_n^{1/4} c_n^{-3/4} + \frac{1}{2}\tilde{c}_n^{1/2} c_n^{-3/2} \right) \frac{1}{n} + \mathcal{O}(n^{-1-\frac{3}{4}\kappa})$$

$$+ \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \hat{\psi}_2(x) \int_x^{\infty} \hat{\psi}_1(y)dydx + \int_0^{n^{-1}} \hat{\psi}_2(x) \int_x^{n^{-1}} \hat{\psi}_1(y)dydx. \qquad (5.40)$$

For notational convenience let us denote the last double integral of this equation again by J. Changing the order of integration, using the asymptotic behavior of $\hat{\psi}_1$ in the Bessel region given by (4.56), and using (5.28), we obtain

$$J = \int_0^{n^{-1}} \hat{\psi}_1(y) \int_0^{y} \hat{\psi}_2(x)\,dxdy$$

$$= -\int_0^{n^{-1}} \frac{\tilde{c}_n^{1/2} n^{1/2}}{\sqrt{y}} J_{\alpha+1}\big(2\tilde{c}_n^{1/2} n\sqrt{y}\big) \int_0^{y} \hat{\psi}_2(x)\,dxdy + \mathcal{O}(n^{-1-\frac{1}{4}}).$$

Changing back the order of integration, using the asymptotic behavior (4.57) of $\hat{\psi}_2$ in the Bessel region, and using the fact that $\int_a^b J_{\alpha+1}(u)du$ is uniformly bounded for $a, b \in [0, \infty]$, we arrive at

$$J = -n^{-1/2} \int_0^{n^{-1}} \hat{\psi}_2(x) \int_{2\tilde{c}_n^{1/2} n\sqrt{x}}^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(v)\, dv dx + \mathcal{O}(n^{-1-\frac{1}{4}})$$

$$= n^{-1} \int_0^{2\tilde{c}_n^{1/2}\sqrt{n}} \left(-2\frac{\alpha}{u} J_\alpha(u) + J_{\alpha+1}(u)\right) \int_u^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(v)\, dv\, du + \mathcal{O}(n^{-1-\frac{1}{4}}). \tag{5.41}$$

Since $\frac{\alpha}{u} J_\alpha(u) = J'_\alpha(u) + J_{\alpha+1}(u)$, see e.g. [Ref. 1, (9.1.27)], we then have from Proposition 5.7,

$$J = -2n^{-1} \int_0^{2\tilde{c}_n^{1/2}\sqrt{n}} J'_\alpha(u) \int_u^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(v)\, dv\, du$$

$$- n^{-1} \int_0^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(u) \int_u^{2\tilde{c}_n^{1/2}\sqrt{n}} J_{\alpha+1}(v)\, dv\, du + \mathcal{O}(n^{-1-\frac{1}{4}})$$

$$= -\frac{3}{2} n^{-1} + \mathcal{O}(n^{-1-\frac{1}{4}}). \tag{5.42}$$

Inserting this into (5.40) and using (4.11) and (4.12) the Proposition is now proven. □

5.2.2. The Double Integrals in the Bulk

Here we will determine the asymptotic behavior (as $n \to \infty$) of the following three double integrals which appear in Proposition 5.11,

$$J_1 \equiv \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_q}}^{\infty} \hat{\phi}_q(y)\, dy\, dx, \quad J_2 \equiv \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_n}}^{\infty} \hat{\psi}_r(y)\, dy\, dx,$$

and

$$J_3 \equiv \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \hat{\psi}_2(x) \int_x^{\infty} \hat{\psi}_1(y)\, dy\, dx,$$

with $p = n + i$ and $q = n + j$ for some fixed integers i, j, and with $r \in \{1, 2\}$. In order to determine the asymptotics we proceed as in the derivation of the asymptotics of the double integral J_3 under Eq. (4.120) in Ref. 7. We will need the following auxiliary results.

Proposition 5.12. *The scalar function*

$$\theta(x) = \frac{1}{2}\int_0^x \sqrt{\frac{1-s}{s}}\,h(s)\,ds, \quad \textit{for } x \in [0,1], \tag{5.43}$$

satisfies the following differential equation,

$$\theta(x) - \frac{1}{m}x\theta'(x) - \pi = -\arccos(2x-1). \tag{5.44}$$

Proof: The proof is similar to the proof of [Ref. 7, Lemma 4.8]. We will need the first and second derivative of θ. From (5.43) we have

$$\theta'(x) = \frac{1}{2}(1-x)^{1/2}x^{-1/2}h(x), \tag{5.45}$$

$$\theta''(x) = -\frac{1}{4}(1-x)^{-1/2}x^{-3/2}(h(x) - 2x(1-x)h'(x)). \tag{5.46}$$

Now, we will obtain a convenient expression for θ'' by deriving a differential equation for h, cf. [Ref. 7, Proposition 6.2]. Since $h(x) = \frac{4m}{2m-1}\,{}_2F_1(1, 1-m, 3/2-m, x)$, it satisfies the following hypergeometric equation (see [Ref. 1, (15.5.1)]),

$$x(1-x)h''(x) + \left(\left(-m+\frac{3}{2}\right) + (m-3)x\right)h'(x) + (m-1)h(x) = 0,$$

which in turn implies that

$$\frac{d}{dx}(x(1-x)h'(x) - [(m-1/2) - (m-1)x]h(x)) = 0.$$

Therefore, the function inside the outer brackets is a constant, which can be determined by letting $x \to 1$. We then obtain the following differential equation for h,

$$x(1-x)h'(x) - [(m-1/2) - (m-1)x]h(x) = -\frac{1}{2}h(1) = -2m. \tag{5.47}$$

Inserting (5.47) into (5.46) we obtain

$$\theta''(x) = -\frac{1}{4}(1-x)^{-1/2}x^{-3/2}(4m - 2(m-1)(1-x)h(x)),$$

which implies, together with (5.45), that

$$\frac{d}{dx}\left(\theta(x) - \frac{1}{m}x\theta'(x)\right) = \frac{(m-1)\theta'(x) - x\theta''(x)}{m} = (1-x)^{-1/2}x^{-1/2}.$$

Therefore,

$$\theta(x) - \frac{1}{m}x\theta'(x) = \int_0^x \frac{dy}{\sqrt{y(1-y)}} = \pi - \arccos(2x-1), \tag{5.48}$$

and the Proposition is proven. □

Proposition 5.13. *Let $p = n + i$ and $q = n + j$ for some fixed integers i, j. Uniformly for $x \in (0, 1 - p^{\kappa-2/3}]$, as $n \to \infty$,*

$$F_q\left(x\frac{\beta_p}{\beta_q}\right) - F_p(x) = -(p-q)\arccos(2x-1) + \mathcal{O}(n^{-\frac{1}{3m}}). \tag{5.49}$$

Proof: The proof of this Proposition is similar to the proof of [Ref. 7, Lemma 4.7]. We write the left hand side of (5.49) as,

$$F_q\left(x\frac{\beta_p}{\beta_q}\right) - F_p(x) = \left[F_q\left(x\frac{\beta_p}{\beta_q}\right) - F_q(x)\right] + [F_q(x) - F_p(x)], \tag{5.50}$$

and we treat each of the terms inside the brackets separately. First, there exists a number $\xi_{n,x}$ between x and $x\frac{\beta_p}{\beta_q}$ such that,

$$F_q\left(x\frac{\beta_p}{\beta_q}\right) - F_q(x) = xF_q'(x)\left(\frac{\beta_p}{\beta_q} - 1\right) + \frac{1}{2}x^2F_q''(\xi_{n,x})\left(\frac{\beta_p}{\beta_q} - 1\right)^2. \tag{5.51}$$

From (5.4), from the fact that $h_q(x) = h(x) + \mathcal{O}(n^{-1/m})$, and from (5.45) we have

$$xF_q'(x) = -qx\theta'(x) + \mathcal{O}(n^{1-1/m}) + \mathcal{O}(n^{2/3-\kappa}).$$

Further, from (5.5) and from the fact that $\xi_{n,x} = x(1 + \mathcal{O}(1/n))$, we obtain,

$$x^2F_q''(\xi_{n,x}) = \mathcal{O}(n^{4/3-\frac{1}{2}\kappa}).$$

Inserting these two equations into (5.51) and using Proposition 5.8 we arrive at

$$F_q\left(x\frac{\beta_p}{\beta_q}\right) - F_q(x) = -(p-q)\frac{1}{m}x\theta'(x) + \mathcal{O}(n^{-1/m}) + \mathcal{O}(n^{-1/3-\kappa}). \tag{5.52}$$

Next, we determine the asymptotic behavior of the second term in (5.50). Note that by (4.15) and (4.6),

$$F_p(x) = p\pi - \frac{p}{2}\int_0^x \sqrt{\frac{1-s}{s}}h_p(s)\,ds + \frac{1}{2}(\alpha+1)\arccos(2x-1) - \frac{\pi}{4},$$

which implies that

$$F_q(x) - F_p(x) = (q-p)\pi + \frac{1}{2}\int_0^x \sqrt{\frac{1-s}{s}}(ph_p(s) - qh_q(s))\,ds.$$

Now,

$$
\begin{aligned}
ph_p(s) - qh_q(s) &= (p-q)h(s) + \sum_{\ell=1}^{m} h_{(\ell)}(s)(p^{1-\ell/m} - q^{1-\ell/m}) + \mathcal{O}(n^{-1/m}) \\
&= (p-q)h(s) + \mathcal{O}(n^{-1/m}), \quad \text{as } n \to \infty,
\end{aligned}
$$

uniformly for $s \in [0, 1]$, so that

$$
F_q(x) - F_p(x) = (p-q)(\theta(x) - \pi) + \mathcal{O}(n^{-1/m}). \tag{5.53}
$$

Inserting Eqs. (5.52) and (5.53) into Eq. (5.50), the relation (5.49) follows from the previous Proposition. □

Asymptotics of J_1. We start with the asymptotic behavior of the double integral J_1. From Eqs. (5.13) and (5.19), and from the asymptotic behavior (4.47) of $\hat{\phi}_p$ in the bulk region, we obtain

$$
\begin{aligned}
J_1 &= \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_q}}^{1-q^{\kappa-\frac{2}{3}}} \hat{\phi}_q(y)\, dy\, dx + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}) \\
&= \sqrt{\frac{2}{\pi}} \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \int_{x\frac{\beta_p}{\beta_q}}^{1-q^{\kappa-\frac{2}{3}}} \hat{\phi}_q(y)\, dy\, dx + \mathcal{O}\big(n^{-1-\frac{3}{4}\kappa}\big).
\end{aligned} \tag{5.54}
$$

Observe that $x\frac{\beta_p}{\beta_q} \in [\frac{1}{2}q^{-1}, 1 - \frac{1}{2}q^{\varepsilon-\frac{2}{3}}]$ if $x \in [p^{-1}, 1 - p^{\kappa-\frac{2}{3}}]$ and n is sufficiently large. By changing the order of integration and using (5.10) we derive the estimate

$$
\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \int_{x\frac{\beta_p}{\beta_q}}^{1-q^{\kappa-\frac{2}{3}}} \mathcal{O}\left(\frac{1}{qy^{3/4}(1-y)^{7/4}}\right) dy\, dx = \mathcal{O}(n^{-1-\frac{3}{2}\kappa}). \tag{5.55}
$$

The asymptotic behavior of $\hat{\phi}_q$ in the bulk region, given by (4.47), together with (5.54) and (5.55), leads to

$$
J_1 = \frac{2}{\pi} \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \int_{x\frac{\beta_p}{\beta_q}}^{1-q^{\kappa-\frac{2}{3}}} \frac{\cos F_q(y)}{y^{1/4}(1-y)^{1/4}}\, dy\, dx + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}).
$$

Integrating by parts the inner integral of this expression and using (5.8) we obtain

$$J_1 = -\frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \frac{\sin F_q\left(x\frac{\beta_p}{\beta_q}\right)}{F_q'\left(x\frac{\beta_p}{\beta_q}\right)\left(x\frac{\beta_p}{\beta_q}\right)^{1/4}\left(1-x\frac{\beta_p}{\beta_q}\right)^{1/4}}dx$$

$$+ \frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}}dx \left.\frac{\sin F_q(y)}{F_q'(y)y^{1/4}(1-y)^{1/4}}\right|_{y=1-q^{\kappa-\frac{2}{3}}}$$

$$- \frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \int_{x\frac{\beta_p}{\beta_q}}^{1-q^{\kappa-\frac{2}{3}}} \mathcal{O}\left(\frac{1}{qy^{3/4}(1-y)^{7/4}}\right) dy\, dx$$

$$+ \mathcal{O}(n^{-1-\frac{3}{4}\kappa}).$$

From Eqs. (5.6), (5.10) and (5.55), we then have

$$J_1 = -\frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \frac{\sin F_q\left(x\frac{\beta_p}{\beta_q}\right)}{F_q'\left(x\frac{\beta_p}{\beta_q}\right)\left(x\frac{\beta_p}{\beta_q}\right)^{1/4}\left(1-x\frac{\beta_p}{\beta_q}\right)^{1/4}}dx$$

$$+\mathcal{O}\left(n^{-1-\frac{3}{4}\kappa}\right). \tag{5.56}$$

Now we will determine a convenient expression for the integrand. Note that, for some $\xi_{n,x}$ between x and $x\frac{\beta_p}{\beta_q}$,

$$\frac{1}{F_q'\left(x\frac{\beta_p}{\beta_q}\right)} = \frac{1}{F_q'(x)}\left[1 + x\frac{F_q''(\xi_{n,x})}{F_q'(x)}\left(\frac{\beta_p}{\beta_q} - 1\right)\right]^{-1}.$$

Since $\xi_{n,x} = x(1+\mathcal{O}(1/n))$, one has by Propositions 5.11 and 5.8

$$x\frac{F_q''(\xi_{n,x})}{F_q'(x)}\left(\frac{\beta_p}{\beta_q} - 1\right) = \mathcal{O}\left(\frac{1}{n(1-x)}\right),$$

so that by (5.4),

$$\frac{1}{F_q'\left(x\frac{\beta_p}{\beta_q}\right)\left(x\frac{\beta_p}{\beta_q}\right)^{1/4}\left(1-x\frac{\beta_p}{\beta_q}\right)^{1/4}} = \frac{1}{F_q'(x)x^{1/4}(1-x)^{1/4}}\left[1+\mathcal{O}\left(\frac{1}{n(1-x)}\right)\right]$$

$$= \frac{-2x^{1/4}}{qh_q(x)(1-x)^{3/4}}\left[1+\mathcal{O}\left(\frac{1}{n(1-x)}\right)\right]. \tag{5.57}$$

Inserting this expression into Eq. (5.56) we arrive at,

$$
\begin{aligned}
J_1 &= \frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{2\cos F_p(x)\sin F_q\left(x\frac{\beta_p}{\beta_q}\right)}{qh_q(x)(1-x)}dx + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}) \\
&= \frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\sin\left(F_q\left(x\frac{\beta_p}{\beta_q}\right)-F_p(x)\right)}{qh_q(x)(1-x)}\,dx \\
&\quad + \frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\sin\left(F_q\left(x\frac{\beta_p}{\beta_q}\right)+F_p(x)\right)}{qh_q(x)(1-x)}\,dx + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}) \\
&\equiv J_1' + J_1'' + (n^{-1-\frac{3}{4}\kappa}).
\end{aligned} \tag{5.58}
$$

It remains to determine the asymptotic behavior of J_1' and J_1''. Using partial integration and using calculations similar to those used in proving (5.8) we can show that

$$J_1'' = \mathcal{O}(n^{-1-\frac{3}{2}\kappa}).$$

From Proposition 5.13 and from $1/h_q(x) = 1/h(x) + \mathcal{O}(n^{-1/m})$, see (4.7), we have uniformly for $x \in (0, 1-p^{\kappa-\frac{2}{3}}]$,

$$\frac{1}{h_q(x)}\sin\left(F_q\left(x\frac{\beta_p}{\beta_q}\right)-F_p(x)\right) = -\frac{1}{h(x)}\sin((p-q)\arccos(2x-1)) + \mathcal{O}(n^{-\frac{1}{3m}}),$$

so that

$$J_1' = -\frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\sin((p-q)\arccos(2x-1))}{qh(x)(1-x)}\,dx + \mathcal{O}(n^{-1-\frac{1}{3m}}\log n).$$

In conclusion we have shown that that there exists $0 < \tau < 1$ such that as $n \to \infty$,

$$J_1 = -\hat{I}(p-q)n^{-1} + \mathcal{O}(n^{-1-\tau}), \tag{5.59}$$

with $\hat{I}$ given by (2.32).

Asymptotics of J_2. Next, we determine the asymptotics of J_2. From Eqs. (5.13) and (5.28), and from the asymptotic behavior of $\hat{\phi}_p$ in the bulk region given by (4.47), we have,

$$
\begin{aligned}
J_2 &= \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \hat{\phi}_p(x)\int_{x\frac{\beta_p}{\beta_n}}^{1-n^{\kappa-\frac{2}{3}}} \hat{\psi}_r(y)\,dy\,dx + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}) \\
&= \sqrt{\frac{2}{\pi}}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}}\int_{x\frac{\beta_p}{\beta_n}}^{1-n^{\kappa-\frac{2}{3}}} \hat{\psi}_r(y)\,dy\,dx + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}).
\end{aligned}
$$

By changing the order of integration and using Eq. (5.10) we obtain the analog of Eq. (5.55),

$$\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \int_{x\frac{\beta_p}{\beta_n}}^{1-n^{\kappa-\frac{2}{3}}} \mathcal{O}\left(\frac{1}{ny^{5/4}(1-y)^{7/4}}\right) dy\, dx = \mathcal{O}(n^{-1-\frac{3}{2}\kappa}). \tag{5.60}$$

Using the asymptotic behavior (4.60) of $\hat{\psi}_r$ in the bulk region we then obtain,

$$\begin{aligned} J_2 &= \frac{(-1)^n \tilde{c}_n^{1/4}}{\sqrt{2}} \frac{2}{\pi} \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \int_{\frac{\beta_p}{\beta_n}x}^{1-n^{\kappa-\frac{2}{3}}} \frac{\cos G_n(y)}{y^{3/4}(1-y)^{1/4}}\, dy\, dx \\ &\quad + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}) \\ &\equiv \frac{(-1)^n \tilde{c}_n^{1/4}}{\sqrt{2}} \hat{J}_2 + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}). \end{aligned} \tag{5.61}$$

Here we have introduced the notation $\hat{J}_2$ for notational convenience. Integrating the inner integral of $\hat{J}_2$ by parts, and using (5.9) we have,

$$\begin{aligned} \hat{J}_2 = &-\frac{2}{\pi} \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \frac{\sin G_n\left(\frac{\beta_p}{\beta_n}x\right)}{G_n'\left(\frac{\beta_p}{\beta_n}x\right)\left(\frac{\beta_p}{\beta_n}x\right)^{3/4}\left(1-\frac{\beta_p}{\beta_n}x\right)^{1/4}} dx \\ &+ \frac{2}{\pi} \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} dx\, \left. \frac{\sin G_n(y)}{G_n'(y) y^{3/4}(1-y)^{1/4}} \right|_{y=1-n^{\kappa-\frac{2}{3}}} \\ &- \frac{2}{\pi} \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \int_{x\frac{\beta_p}{\beta_n}}^{1-n^{\kappa-\frac{2}{3}}} \mathcal{O}\left(\frac{1}{ny^{5/4}(1-y)^{7/4}}\right) dy\, dx \\ &+ \mathcal{O}(n^{-1-\frac{3}{4}\kappa}). \end{aligned}$$

From (5.7), (5.10) and (5.60) we arrive at,

$$\hat{J}_2 = -\frac{2}{\pi} \int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\cos F_p(x)}{x^{1/4}(1-x)^{1/4}} \frac{\sin G_n\left(\frac{\beta_p}{\beta_n}x\right)}{G_n'\left(\frac{\beta_p}{\beta_n}x\right)\left(\frac{\beta_p}{\beta_n}x\right)^{3/4}\left(1-\frac{\beta_p}{\beta_n}x\right)^{1/4}}\, dx + \mathcal{O}\left(n^{-1-\frac{3}{4}\kappa}\right).$$

As in (5.57) we are led to

$$\frac{1}{G_n'\left(\frac{\beta_p}{\beta_n}x\right)\left(\frac{\beta_p}{\beta_n}x\right)^{3/4}\left(1-\frac{\beta_p}{\beta_n}x\right)^{1/4}} = \frac{-2}{nh_n(x)x^{1/4}(1-x)^{3/4}} \left[1 + \mathcal{O}\left(\frac{1}{n(1-x)}\right)\right],$$

which yields

$$\hat{J}_2 = \frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{2\cos F_p(x)\sin G_n\left(\frac{\beta_p}{\beta_n}x\right)}{nh_n(x)x^{1/2}(1-x)}\,dx + \mathcal{O}\left(n^{-1-\frac{3}{4}\kappa}\right)$$

$$= \frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\sin\left(G_n\left(\frac{\beta_p}{\beta_n}x\right) - F_p(x)\right)}{nh_n(x)x^{1/2}(1-x)}\,dx$$

$$+ \frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\sin\left(G_n\left(\frac{\beta_p}{\beta_n}x\right) + F_p(x)\right)}{nh_n(x)x^{1/2}(1-x)}\,dx\mathcal{O} + (n^{-1-\frac{3}{4}\kappa})$$

$$\equiv \hat{J}_2' + \hat{J}_2'' + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}).$$

As before one can show that $\hat{J}_2'' = \mathcal{O}(n^{-1-\frac{3}{2}\kappa})$. We will now determine the asymptotic behavior of $\hat{J}_2'$. Note that by Proposition 5.13,

$$G_n\left(\frac{\beta_p}{\beta_n}x\right) - F_p(x) = F_n\left(\frac{\beta_p}{\beta_n}x\right) - F_p(x) - \frac{1}{2}\arccos\left(2\frac{\beta_p}{\beta_n}x - 1\right)$$

$$= F_n\left(\frac{\beta_p}{\beta_n}x\right) - F_p(x) - \frac{1}{2}\arccos(2x-1) + \mathcal{O}\left(\frac{x^{1/2}}{n(1-x)^{1/2}}\right)$$

$$= -\left(p - n + \frac{1}{2}\right)\arccos(2x-1) + \mathcal{O}\left(n^{-\frac{1}{3m}}\right), \tag{5.62}$$

so that uniformly for $x \in (0, 1 - p^{\kappa-\frac{2}{3}}]$,

$$\frac{1}{h_n(x)}\sin\left(G_n\left(\frac{\beta_p}{\beta_n}x\right) - F_p(x)\right) = -\frac{1}{h(x)}\sin\left(\left(p - n + \frac{1}{2}\right)\arccos(2x-1)\right)$$
$$+\mathcal{O}\left(n^{-\frac{1}{3m}}\right).$$

Therefore,

$$\hat{J}_2' = -\frac{2}{\pi}\int_{p^{-1}}^{1-p^{\kappa-\frac{2}{3}}} \frac{\sin\left(\left(p-n+\frac{1}{2}\right)\arccos(2x-1)\right)}{nh(x)x^{1/2}(1-x)}\,dx + \mathcal{O}\left(n^{-1-\frac{1}{3m}}\log n\right).$$

Using (4.12) we then have shown that there exists $0 < \tau < 1$ such that as $n \to \infty$,

$$J_2 = -(-1)^n\sqrt{\frac{m}{2m-1}}I(p-n+1)n^{-1} + \mathcal{O}(n^{-1-\tau}), \tag{5.63}$$

with I given by (2.33).

Asymptotics of J_3. Finally, we will determine the asymptotic behavior of the double integral J_3. From Eqs. (5.20) and (5.28), and from the asymptotic behavior of $\hat{\psi}_2$ in the bulk region, given by (4.60), we have,

$$J_3 = \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \hat{\psi}_2(x) \int_x^{1-n^{\kappa-\frac{2}{3}}} \hat{\psi}_1(y)\, dy\, dx + \mathcal{O}(n^{-1-\frac{3}{4}\kappa})$$
$$= \frac{(-1)^n \tilde{c}_n^{1/4}}{\sqrt{\pi}} \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \frac{\cos G_n(x)}{x^{3/4}(1-x)^{1/4}} \int_x^{1-n^{\kappa-\frac{2}{3}}} \hat{\psi}_1(y)\, dy\, dx + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}).$$

Now, by changing the order of integration, using the asymptotic behavior (4.60) of $\hat{\psi}_1$ in the bulk region, and using Eq. (5.11), we arrive at

$$J_3 = \frac{(-1)^n \tilde{c}_n^{1/4}}{\sqrt{\pi}} \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \hat{\psi}_1(y) \int_{n^{-1}}^{y} \frac{\cos G_n(x)}{x^{3/4}(1-x)^{1/4}}\, dx\, dy + \mathcal{O}(n^{-1-\frac{3}{4}\kappa})$$
$$= \frac{\tilde{c}_n^{1/2}}{\pi} \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \frac{\cos G_n(y)}{y^{3/4}(1-y)^{1/4}} \int_{n^{-1}}^{y} \frac{\cos G_n(x)}{x^{3/4}(1-x)^{1/4}}\, dx\, dy + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}).$$

Integrating by parts the inner integral and using (5.9) we then obtain,

$$J_3 = \frac{\tilde{c}_n^{1/2}}{\pi} \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \frac{\cos G_n(y) \sin G_n(y)}{G_n'(y) y^{3/2}(1-y)^{1/2}}\, dy$$
$$- \frac{\tilde{c}_n^{1/2}}{\pi} \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \frac{\cos G_n(y)}{y^{3/4}(1-y)^{1/4}}\, dy \left. \frac{\sin G_n(x)}{G_n'(x) x^{3/4}(1-x)^{1/4}} \right|_{x=n^{-1}}$$
$$- \frac{\tilde{c}_n^{1/2}}{\pi} \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \frac{\cos G_n(y)}{y^{3/4}(1-y)^{1/4}} \int_{n^{-1}}^{y} \mathcal{O}\left(\frac{1}{n x^{5/4}(1-x)^{7/4}}\right) dx\, dy$$
$$+ \mathcal{O}(n^{-1-\frac{3}{4}\kappa}).$$

From (5.7), (5.11) and from the fact that

$$\int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \frac{\cos G_n(y)}{y^{3/4}(1-y)^{1/4}} \int_{n^{-1}}^{y} \mathcal{O}\left(\frac{1}{n x^{5/4}(1-x)^{7/4}}\right) dx\, dy = \mathcal{O}(n^{-1-\frac{3}{4}\kappa}),$$

which follows from changing the order of integration together with Eq. (5.11), we then obtain,

$$J_3 = \frac{\tilde{c}_n^{1/2}}{2\pi} \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \frac{\sin(2G_n(y))}{G_n'(y) y^{3/2}(1-y)^{1/2}}\, dy + \mathcal{O}(n^{-1-\frac{3}{4}\kappa}). \tag{5.64}$$

Integrating by parts once more we have,

$$J_3 = \frac{\tilde{c}_n^{1/2}}{2\pi} \int_{n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} \frac{\cos(2G_n(y))}{G_n'(y)y^{3/4}(1-y)^{1/4}} \left(\frac{1}{G_n'(y)y^{3/4}(1-y)^{1/4}} \right)' dy$$

$$- \frac{\tilde{c}_n^{1/2}}{4\pi} \cos(2G_n(y)) \left(\frac{1}{G_n'(y)y^{3/4}(1-y)^{1/4}} \right)^2 \Bigg|_{y=n^{-1}}^{1-n^{\kappa-\frac{2}{3}}} + (n^{-1-\frac{3}{4}\kappa}).$$

Using (5.7) and (5.9) we finally arrive at,

$$J_3 = \mathcal{O}(n^{-1-\frac{3}{4}\kappa}), \quad \text{as } n \to \infty. \tag{5.65}$$

5.2.3. The Result

Lemma 5.14. *Let $p = n + i$ and $q = n + j$ with i, j some fixed integers and let $r \in \{1, 2\}$. There exists $0 < \tau = \tau(m, \alpha) < 1$ such that as $n \to \infty$,*

$$\int_0^\infty \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_q}}^\infty \hat{\phi}_q(y)\, dy\, dx = \left(\frac{1}{2m} - \hat{I}(p-q) + \mathcal{O}(n^{-\tau}) \right) \frac{1}{n}, \tag{5.66}$$

$$\int_0^\infty \hat{\phi}_p(x) \int_{x\frac{\beta_p}{\beta_n}}^\infty \hat{\psi}_r(y)\, dy\, dx = (-1)^n \sqrt{\frac{m}{2m-1}} \left(\frac{1}{2m} - I(p-n+1) + \mathcal{O}(n^{-\tau}) \right) \frac{1}{n}, \tag{5.67}$$

and

$$\int_0^\infty \hat{\psi}_2(x) \int_x^\infty \hat{\psi}_1(y)\, dy\, dx = \left(-\frac{3}{2} + \frac{(-1)^n}{\sqrt{2m-1}} + \frac{1}{2}\frac{1}{2m-1} + \mathcal{O}(n^{-\tau}) \right) \frac{1}{n}. \tag{5.68}$$

Proof: The Lemma is immediate from Proposition 5.11 and from Eqs. (5.59), (5.63) and (5.65). □

5.3. Asymptotics of the Matrix *B*

Let $p = n + i$ and $q = n + j$ with i, j some fixed integers and let $r \in \{1, 2\}$. From Eqs. (5.1)–(5.3), from Lemmas 5.5, 5.7 and 5.14, and from Proposition 5.8, it is immediate that there exists $0 < \tau < 1$ such that as $n \to \infty$, n even,

$$\langle \varepsilon\phi_q, \phi_p \rangle = \frac{\beta_n}{n}(\hat{I}(p-q) + \mathcal{O}(n^{-\tau})), \tag{5.69}$$

$$\langle \varepsilon\psi_r, \phi_p \rangle = \frac{\beta_n}{n}\left(\sqrt{\frac{m}{2m-1}} I(p-n+1) + \frac{(-1)^r}{2\sqrt{m}} + \mathcal{O}(n^{-\tau})\right), \tag{5.70}$$

$$\langle \varepsilon\psi_1, \psi_2 \rangle = \frac{\beta_n}{n}\left(1 - \frac{1}{\sqrt{2m-1}} + \mathcal{O}(n^{-\tau})\right). \tag{5.71}$$

These equations prove Lemma 2.6.

6. PROOF OF THE MAIN RESULTS

Based on the results of the previous sections we will now prove our main results stated in the Introduction to this paper. Recall that the strategy of the proofs was outlined in Remark 2.16. We will treat the different spectral regions (bulk, hard and soft edge) each in a seperate subsection. Full proofs are provided for the hard edge which has no analogue in the Hermite case. For the soft edge and the bulk we do not repeat arguments already presented in Refs. 7, 8.

6.1. The Hard Edge of the Spectrum

Proof of Theorem 1.1(i): This result for $\beta = 2$ has been proven by one of the authors in [Ref. 23, Theorem 2.8(c)], see also Proposition 6.1 below. □

In order to prove Theorem 1.1 for $\beta = 1, 4$ we proceed as in the proof of [Ref. 8, Theorem 1.1]. We need the following six auxiliary propositions (Propositions 6.1–6.6).

Proposition 6.1. *Let $k, j \in \mathbb{N}$. As $n \to \infty$, uniformly for ξ, η in bounded subsets of $(0, \infty)$,*

$$\frac{\partial^{k+j}}{\partial\xi^k \partial\eta^j}\left[\frac{1}{\nu_n^2} K_n\left(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\right)\right] = \frac{\partial^{k+j}}{\partial\xi^k \partial\eta^j} K_J(\xi, \eta) + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}-k}\eta^{\frac{\alpha}{2}-j}}{n}\right). \tag{6.1}$$

Proof: For the sake of brevity, we introduce the following notation,

$$z_n = \frac{z}{4\tilde{c}_n n^2}, \quad \tilde{z}_n = 2(-\tilde{f}_n(z_n))^{1/2}, \tag{6.2}$$

$$\chi_{1,n}(z) = z^{-\alpha/2}\tilde{z}_n J'_\alpha(\tilde{z}_n), \quad \chi_1(z) = z^{-\alpha/2} z^{1/2} J'_\alpha(z^{1/2}), \quad \hat{\chi}_{1,n} = \chi_{1,n} - \chi_1, \tag{6.3}$$

$$\chi_{2,n}(z) = z^{-\alpha/2} J_\alpha(\tilde{z}_n), \quad \chi_2(z) = z^{-\alpha/2} J_\alpha(z^{1/2}), \quad \hat{\chi}_{2,n} = \chi_{2,n} - \chi_2. \tag{6.4}$$

With this notation we obtain from [Ref. 23, (6.1), (6.4) and (6.5)]

$$\begin{aligned}
&\xi^{-\frac{\alpha}{2}}\eta^{-\frac{\alpha}{2}}\left(\frac{1}{\nu_n^2}K_n\big(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\big) - K_J(\xi,\eta)\right)\\
&= \frac{1}{2(\xi-\eta)}\big(\chi_{1,n}(\eta) \quad \chi_{2,n}(\eta)\big)\begin{pmatrix}\chi_{2,n}(\xi)\\ -\chi_{1,n}(\xi)\end{pmatrix} - \frac{1}{2(\xi-\eta)}\big(\chi_1(\eta) \quad \chi_2(\eta)\big)\begin{pmatrix}\chi_2(\xi)\\ -\chi_1(\xi)\end{pmatrix}\\
&\quad + \frac{1}{2\pi i(\xi-\eta)}\big(\pi i\chi_{1,n}(\eta) \quad \chi_{2,n}(\eta)\big)\big(L_n^{-1}(\eta_n)L_n(\xi_n) - I\big)\begin{pmatrix}\chi_{2,n}(\xi)\\ -\pi i\chi_{1,n}(\xi)\end{pmatrix}\\
&= \left(\frac{\hat{\chi}_{1,n}(\eta) - \hat{\chi}_{1,n}(\xi)}{2(\xi-\eta)} \quad \frac{\hat{\chi}_{2,n}(\eta) - \hat{\chi}_{2,n}(\xi)}{2(\xi-\eta)}\right)\begin{pmatrix}\chi_{2,n}(\xi)\\ -\chi_{1,n}(\xi)\end{pmatrix}\\
&\quad + \left(\frac{\chi_1(\eta) - \chi_1(\xi)}{2(\xi-\eta)} \quad \frac{\chi_2(\eta) - \chi_2(\xi)}{2(\xi-\eta)}\right)\begin{pmatrix}\hat{\chi}_{2,n}(\xi)\\ -\hat{\chi}_{1,n}(\xi)\end{pmatrix}\\
&\quad + \frac{1}{2\pi i(\xi-\eta)}\big(\pi i\chi_{1,n}(\eta) \quad \chi_{2,n}(\eta)\big)\big(L_n^{-1}(\eta_n)L_n(\xi_n) - I\big)\begin{pmatrix}\chi_{2,n}(\xi)\\ -\pi i\chi_{1,n}(\xi)\end{pmatrix},
\end{aligned} \tag{6.5}$$

where L_n is the 2×2 matrix valued function defined in [Ref. 23, Lemma 6.1]. We will now denote the first term of the right hand side of Eq. (6.5) by $H_{n,1}(\xi, \eta)$, the second term by $H_{n,2}(\xi, \eta)$, and the third term by $H_{n,3}(\xi, \eta)$.

Observe that it is sufficient to show that the following estimates hold as $n \to \infty$, uniformly for ξ, η in bounded subsets of $(0, \infty)$,

$$\frac{\partial^{k+j}}{\partial\xi^k \partial\eta^j} H_{n,i}(\xi, \eta) = \mathcal{O}(1/n), \quad i = 1, 2, 3. \tag{6.6}$$

Since $z^{-\alpha} J_\alpha(z)$ is even and entire [Ref. 1, (9.1.10)] it follows that χ_1 and χ_2 are also entire. Further, from the form (4.8) of $\tilde{f}_n$ we have that $\chi_{1,n}(z)$ and $\chi_{2,n}(z)$

(and hence also $\hat{\chi}_{1,n}$ and $\hat{\chi}_{2,n}$) are analytic for z in compact subsets of $\mathbb{C}$ and n sufficiently large, and that $\hat{\chi}_{i,n}(z) = \chi_{i,n}(z) - \chi_i(z) = \mathcal{O}(1/n^2)$, for $i = 1, 2$, as $n \to \infty$, uniformly for z in compact subsets of $\mathbb{C}$. Using the above properties we observe for $i = 1, 2$ and $\ell_1, \ell_2 \in \mathbb{N}$ that all derivatives

$$\frac{\partial^{\ell_1+\ell_2}}{\partial\xi^{\ell_1}\partial\eta^{\ell_2}} \frac{\chi_i(\xi) - \chi_i(\eta)}{\xi - \eta}, \quad \text{remain bounded for } \xi, \eta \text{ in compact subsets of } \mathbb{C}, \tag{6.7}$$

and that,

$$\frac{\partial^{\ell_1+\ell_2}}{\partial\xi^{\ell_1}\partial\eta^{\ell_2}} \frac{\hat{\chi}_{i,n}(\xi) - \hat{\chi}_{i,n}(\eta)}{\xi - \eta} = \mathcal{O}(1/n^2), \tag{6.8}$$

$$\frac{\partial^{\ell_1}}{\partial\xi^{\ell_1}} \hat{\chi}_{i,n}(\xi) = \mathcal{O}(1/n^2), \quad \frac{\partial^{\ell_1}}{\partial\xi^{\ell_1}} \chi_{i,n}(\xi) = \mathcal{O}(1), \tag{6.9}$$

as $n \to \infty$, uniformly for ξ, η in compact subsets of $\mathbb{C}$. From (6.7)–(6.9) it now follows that (6.6) holds for $i = 1, 2$.

As in the proof of [Ref. 23, Lemma 6.1] one can show, by writing $L_n^{-1}(\eta_n)L_n(\xi_n) - I$ as a contour integral, that

$$\frac{\partial^{\ell_1+\ell_2}}{\partial\xi^{\ell_1}\partial\eta^{\ell_2}} \frac{L_n^{-1}(\eta_n)L_n(\xi_n) - I}{\xi - \eta} = \mathcal{O}(1/n),$$

as $n \to \infty$, uniformly for ξ, η in bounded subsets of $(0, \infty)$. This together with (6.9) then proves (6.6) for $i = 3$, as well. Hence, the Proposition is proven. □

Proposition 6.2. *As $n \to \infty$, uniformly for ξ, η in bounded subsets of $(0, \infty)$,*

$$\int_0^{\xi} \frac{1}{\nu_n^2} K_n\big(\tilde{s}^{(n)}, \tilde{\eta}^{(n)}\big)\, ds = \int_0^{\xi} K_J(s, \eta)\, ds + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}+1}\eta^{\frac{\alpha}{2}}}{n}\right), \tag{6.10}$$

$$\int_{\xi}^{\eta} \frac{1}{\nu_n^2} K_n\big(\tilde{s}^{(n)}, \tilde{\eta}^{(n)}\big)\, ds = \int_{\xi}^{\eta} K_J(s, \eta)\, ds + \mathcal{O}\left(\frac{\eta^{\frac{\alpha}{2}}}{n}\right). \tag{6.11}$$

Proof: This is immediate from Proposition 6.1. □

Proposition 6.3. *There exists* $0 < \tau = \tau(m, \alpha) < 1$ *such that as (even)* $n \to \infty$,

$$\varepsilon\Phi_1(+\infty) = \frac{1}{2}\sqrt{\frac{\beta_n}{n}}\left[\frac{1}{\sqrt{m}}\,\mathbf{a} - \mathbf{e} + \mathcal{O}(n^{-\tau})\right], \tag{6.12}$$

$$\varepsilon\Phi_2(+\infty) = \frac{1}{2}\sqrt{\frac{\beta_n}{n}}\left[\frac{1}{\sqrt{m}}\,\mathbf{a} + \mathbf{e} + \mathcal{O}(n^{-\tau})\right], \tag{6.13}$$

where **a** *and* **e** *are m-dimensional row vectors given by,*

$$\mathbf{a} = \left(1, \dots, 1, \sqrt{\frac{m}{2m-1}}\right), \quad \mathbf{e} = (0, \dots, 0, 1). \tag{6.14}$$

Proof: Fix $j \in \mathbb{Z}$ and let $r = 1, 2$. From Lemma 5.5 and Proposition 5.8 we have

$$\int_0^\infty \phi_{n+j}(x)\,dx = \sqrt{\beta_{n+j}}\int_0^\infty \hat{\phi}_{n+j}(x)\,dx = \sqrt{\frac{\beta_{n+j}}{n+j}}\left(\frac{1}{\sqrt{m}} + \mathcal{O}(n^{-\tau})\right)$$
$$= \sqrt{\frac{\beta_n}{n}}\left(\frac{1}{\sqrt{m}} + \mathcal{O}(n^{-\tau})\right),$$

and from Lemma 5.7 we have for n even,

$$\int_0^\infty \psi_r(x)\,dx = \sqrt{\beta_n}\int_0^\infty \hat{\psi}_r(x)\,dx = \sqrt{\frac{\beta_n}{n}}\left(\frac{1}{\sqrt{2m-1}} + (-1)^r + \mathcal{O}(n^{-\tau})\right),$$

for some $0 < \tau < 1$. Since $\varepsilon\Phi_r(+\infty) = \frac{1}{2}\int_0^\infty \Phi_r(x)\,dx$ this proves the Proposition. □

Proposition 6.4. *Uniformly for* ξ *in bounded subsets of* $(0, \infty)$, *as* $n \to \infty$

$$\frac{1}{\nu_n^2}\Phi_1(\tilde{\xi}^{(n)}) = -\frac{1}{2}\sqrt{\frac{\beta_n}{n}}\left[\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}}\cdot\mathbf{e} + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}}}{n}\right)\right], \tag{6.15}$$

$$\frac{1}{\nu_n^2}\Phi_2(\tilde{\xi}^{(n)}) = -\frac{1}{2}\sqrt{\frac{\beta_n}{n}}\left[\left(\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}} - \frac{2\alpha}{\xi}J_\alpha(\sqrt{\xi})\right)\cdot\mathbf{e} + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}}}{n}\right)\right]. \tag{6.16}$$

Proof: The Proposition follows from Eqs. (4.56) and (4.57), and from the fact that for every $j \in \mathbb{Z}$,

$$\frac{1}{\nu_n^2}\phi_{n+j}\big(\tilde{\xi}^{(n)}\big) = \frac{\beta_n}{4\tilde{c}_n n^2}\phi_{n+j}\left(\beta_{n+j}\frac{\xi}{4\tilde{c}_n n^2}\frac{\beta_n}{\beta_{n+j}}\right)$$

$$= \frac{\beta_n}{4\tilde{c}_n n^2}\frac{1}{\sqrt{\beta_{n+j}}}\hat{\phi}_{n+j}\left(\frac{\xi}{4\tilde{c}_n n^2}\frac{\beta_n}{\beta_{n+j}}\right) = \mathcal{O}\left(\sqrt{\frac{\beta_n}{n}}\frac{\xi^{\frac{\alpha}{2}}}{n}\right).$$

In the last equality we have used (4.46). □

Proposition 6.5. *Uniformly for ξ, η in bounded subsets of $(0,\infty)$, as $n \to \infty$*

$$\int_0^{\tilde{\eta}^{(n)}} \Phi_1(s)\,ds = -\sqrt{\frac{\beta_n}{n}}\left[\int_0^{\sqrt{\eta}} J_{\alpha+1}(s)\,ds \cdot \mathbf{e} + \mathcal{O}\left(\frac{\eta^{\frac{\alpha}{2}+1}}{n}\right)\right], \tag{6.17}$$

$$\int_0^{\tilde{\eta}^{(n)}} \Phi_2(s)\,ds = -\sqrt{\frac{\beta_n}{n}}\left[\int_0^{\sqrt{\eta}} \left(J_{\alpha+1}(s) - \frac{2\alpha}{s}J_\alpha(s)\right) ds \cdot \mathbf{e} + \mathcal{O}\left(\frac{\eta^{\frac{\alpha}{2}+1}}{n}\right)\right], \tag{6.18}$$

$$\int_{\tilde{\xi}^{(n)}}^{\tilde{\eta}^{(n)}} \Phi_1(s)\,ds = -\sqrt{\frac{\beta_n}{n}}\left[\int_{\sqrt{\xi}}^{\sqrt{\eta}} J_{\alpha+1}(s)\,ds \cdot \mathbf{e} + \mathcal{O}\left(\frac{1}{n}\right)\right]. \tag{6.19}$$

Proof: This is immediate from Proposition 6.4. □

Proposition 6.6. *There exists $0 < \tau = \tau(m,\alpha) < 1$ such that, uniformly for η in bounded subsets of $(0,\infty)$, as $n \to \infty$, n even,*

$$\int_0^{\tilde{\eta}^{(n)}} \Phi_1(s)\,ds - \varepsilon\Phi_1(+\infty) + \varepsilon\Phi_2(+\infty) = \sqrt{\frac{\beta_n}{n}}\left[\int_{\sqrt{\eta}}^{\infty} J_{\alpha+1}(s)\,ds \cdot \mathbf{e} + \mathcal{O}(n^{-\tau})\right],$$

$$\int_0^{\tilde{\eta}^{(n)}} \Phi_2(s)\,ds - \varepsilon\Phi_2(+\infty) + \varepsilon\Phi_1(+\infty) \tag{6.20}$$

$$= \sqrt{\frac{\beta_n}{n}}\left[\int_{\sqrt{\eta}}^{\infty} \left(J_{\alpha+1}(s) - \frac{2\alpha}{s}J_\alpha(s)\right) ds \cdot \mathbf{e} + \mathcal{O}(n^{-\tau})\right]. \tag{6.21}$$

Proof: This follows from Eqs. (6.17) and (6.18), from Proposition 6.3, and from the facts that $\int_0^\infty J_{\alpha+1}(s)\,ds = 1$ and $\int_0^\infty \frac{\alpha}{s} J_\alpha(s)\,ds = 1$, see [Ref. 1, (11.4.16) and (11.4.17)]. □

Now we have the necessary ingredients to prove Theorem 1.1 for the cases $\beta = 1, 4$.

Proof of Theorem 1.1(ii): The (1, 1)- and (2, 2)-entry: By (1.11), (1.10) and (2.51), we have

$$\frac{2}{\nu_n^2}\left[K^{(\nu_n)}_{\frac{n}{2},4}\big(\tilde\xi^{(n)},\tilde\eta^{(n)}\big)\right]_{11} = \frac{1}{\nu_n^2} S_{\frac{n}{2},4}\big(\tilde\xi^{(n)},\tilde\eta^{(n)}\big)$$
$$= \frac{1}{\nu_n^2} K_n\big(\tilde\xi^{(n)},\tilde\eta^{(n)}\big) - \frac{1}{\nu_n^2}\Phi_2\big(\tilde\xi^{(n)}\big)A_{21}\int_0^{\tilde\eta^{(n)}}\Phi_1(s)^t\,ds - \frac{1}{\nu_n^2}\Phi_2(\tilde\xi^{(n)})G_{11}\int_0^{\tilde\eta^{(n)}}\Phi_2(s)^t\,ds.$$

The asymptotics of the first term of the right hand side of the latter equation have been determined in part (i) of the theorem. From (6.16), (6.17), and the facts that $\mathbf{e}A_{21}\mathbf{e}^t = -\frac{1}{2}\frac{n}{\beta_n}$ (which follows from Eq. (2.19)) and $A_{21} = \mathcal{O}(\frac{n}{\beta_n})$ (see Lemma 2.5), we obtain

$$\frac{1}{\nu_n^2}\Phi_2\big(\tilde\xi^{(n)}\big)A_{21}\int_0^{\tilde\eta^{(n)}}\Phi_1(s)^t\,ds$$
$$= \frac{1}{2}\left[\left(\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}} - \frac{2\alpha}{\xi}J_\alpha(\sqrt{\xi})\right)\cdot\mathbf{e} + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}}}{n}\right)\right]\frac{\beta_n}{n}A_{21}$$
$$\times\left[\int_0^{\sqrt{\eta}} J_{\alpha+1}(s)\,ds\cdot\mathbf{e}^t + \mathcal{O}\left(\frac{\eta^{\frac{\alpha}{2}+1}}{n}\right)\right]$$
$$= -\frac{1}{4}\left(\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}} - \frac{2\alpha}{\xi}J_\alpha(\sqrt{\xi})\right)\int_0^{\sqrt{\eta}} J_{\alpha+1}(s)\,ds + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}-1}\eta^{\frac{\alpha}{2}+1}}{n}\right).$$

From (6.16), (6.18), and the facts that $\mathbf{e}G_{11}\mathbf{e}^t = 0$ (which follows from the skew symmetry of G_{11}, see Lemma 2.10) and $G_{11} = \mathcal{O}(\frac{n}{\beta_n})$ (see Corollary 2.13), we have

$$\frac{1}{\nu_n^2}\Phi_2\big(\tilde{\xi}^{(n)}\big)G_{11}\int_0^{\tilde{\eta}^{(n)}}\Phi_2(s)^t ds$$
$$= \left[\mathcal{O}(\xi^{\frac{\alpha}{2}-1})\cdot\mathbf{e}+\mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}}}{n}\right)\right]\frac{\beta_n}{n}G_{11}\left[\mathcal{O}(\eta^{\frac{\alpha}{2}})\cdot\mathbf{e}^t+\mathcal{O}\left(\frac{\eta^{\frac{\alpha}{2}+1}}{n}\right)\right]$$
$$= \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}-1}\eta^{\frac{\alpha}{2}}}{n}\right).$$

We conclude that

$$\frac{2}{\nu_n^2}\big[K^{(\nu_n)}_{\frac{n}{2},4}\big(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\big)\big]_{11}$$
$$= K_J(\xi,\eta)+\frac{1}{4}\left(\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}}-\frac{2\alpha}{\xi}J_\alpha(\sqrt{\xi})\right)\int_0^{\sqrt{\eta}}J_{\alpha+1}(s)\,ds$$
$$+\mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}-1}\eta^{\frac{\alpha}{2}}}{n}\right). \tag{6.22}$$

The (1, 2)-entry: Again by (2.51) we have,

$$\left(-\frac{\partial}{\partial y}S_{\frac{n}{2},4}\right)(x,y) = -\frac{\partial}{\partial y}K_n(x,y)+\Phi_2(x)A_{21}\Phi_1(y)^t+\Phi_2(x)G_{11}\Phi_2(y)^t.$$

As for the (1, 1)- and (2, 2)-entry, we obtain from Propositions 6.1 and 6.4,

$$\frac{2}{\nu_n^2}\big[K^{(\nu_n)}_{\frac{n}{2},4}\big(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\big)\big]_{12} = \frac{1}{\nu_n^4}\left(-\frac{\partial}{\partial y}S_{\frac{n}{2},4}\right)\big(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\big)$$
$$= -\frac{\partial}{\partial\eta}\left(\frac{1}{\nu_n^2}K_n\big(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\big)\right)+\frac{1}{\nu_n^4}\Phi_2\big(\tilde{\xi}^{(n)}\big)A_{21}\Phi_1\big(\tilde{\eta}^{(n)}\big)^t+\frac{1}{\nu_n^4}\Phi_2(\tilde{\xi}^{(n)})G_{11}\Phi_2(\tilde{\eta}^{(n)})^t$$
$$= -\frac{\partial}{\partial\eta}K_J(\xi,\eta)-\frac{1}{8}\left(\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}}-\frac{2\alpha}{\xi}J_\alpha(\sqrt{\xi})\right)\frac{J_{\alpha+1}(\sqrt{\eta})}{\sqrt{\eta}}+\mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}-1}\eta^{\frac{\alpha}{2}-1}}{n}\right). \tag{6.23}$$

The (2, 1)-entry: Using relation $(\varepsilon S_{\frac{n}{2},4})(x,y) = \int_0^x S_{\frac{n}{2},4}(s,y)\,ds$ of Proposition 2.1, we obtain from (2.51),

$$(\varepsilon S_{\frac{n}{2},4})(x,y) = \int_0^x K_n(s,y)\,ds - \int_0^x \Phi_2(s)\,ds\,A_{21}\int_0^y \Phi_1(s)^t ds \\ - \int_0^x \Phi_2(s)\,ds\,G_{11}\int_0^y \Phi_2(s)^t ds. \tag{6.24}$$

Therefore, we obtain from (6.10), (6.17) and (6.18) in the same way as before,

$$\begin{aligned}\frac{2}{\nu_n^2}\big[K^{(\nu_n)}_{\frac{n}{2},4}\big(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\big)\big]_{21} &= (\varepsilon S_{\frac{n}{2},4})\big(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\big) \\ &= \int_0^{\xi} \frac{1}{\nu_n^2} K_n(\tilde{s}^{(n)},\tilde{\eta}^{(n)})\,ds - \int_0^{\tilde{\xi}^{(n)}} \Phi_2(s)\,ds\,A_{21}\int_0^{\tilde{\eta}^{(n)}} \Phi_1(s)^t ds \\ &\quad - \int_0^{\tilde{\xi}^{(n)}} \Phi_2(s)\,ds\,G_{11}\int_0^{\tilde{\eta}^{(n)}} \Phi_2(s)^t ds \\ &= \int_0^{\xi} K_J(s,\eta)\,ds + \frac{1}{2}\int_0^{\sqrt{\xi}} \left(J_{\alpha+1}(s) - \frac{2\alpha}{s}J_\alpha(s)\right) ds \\ &\quad \times \int_0^{\sqrt{\eta}} J_{\alpha+1}(s)\,ds + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}}\eta^{\frac{\alpha}{2}}}{n}\right).\end{aligned} \tag{6.25}$$

This concludes the proof of the second part of the Theorem. □

Proof of Theorem 1.1(iii): THE (1, 1)- AND (2, 2)-ENTRY: Using $\varepsilon\Phi_1(+\infty) = \mathcal{O}(\sqrt{\frac{\beta_n}{n}}) = \varepsilon\Phi_2(+\infty)$ (see Proposition 6.3), $A_{12} = \mathcal{O}(\frac{n}{\beta_n})$ (see Lemma 2.5), $\hat{C}_{22}^{-1} = \mathcal{O}(1)$ (see Corollary 2.12) and (6.15), we obtain the following estimate for the last term in (2.53)

$$\frac{1}{\nu_n^2}\Phi_1\big(\tilde{\xi}^{(n)}\big)A_{12}\hat{C}_{22}^{-1}\left[\mathcal{O}(n^{-\tau})\varepsilon\Phi_1(+\infty)^t + \mathcal{O}(n^{-\tau})\varepsilon\Phi_2(+\infty)^t\right] = \mathcal{O}(\xi^{\frac{\alpha}{2}}n^{-\tau}).$$

By (1.11), (1.9), (2.53), Proposition 6.1, Eq. (6.15), and Proposition 6.6. we then derive in the same way as before (note that also $\hat{G}_{11}$ is skew symmetric, see Lemma

2.8(ii), and that also $\mathbf{e}A_{12}\mathbf{e}^t = -\frac{1}{2}\frac{n}{\beta_n}$)

$$\begin{aligned}
\frac{1}{\nu_n^2}\big[K_{n,1}^{(\nu_n)}\big(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\big)\big]_{11} &= \frac{1}{\nu_n^2} S_{n,1}\big(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\big) \\
&= \frac{1}{\nu_n^2} K_n\big(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\big) \\
&\quad - \frac{1}{\nu_n^2}\Phi_1\big(\tilde{\xi}^{(n)}\big)A_{12}\left(\int_0^{\tilde{\eta}^{(n)}} \Phi_2(s)^t ds - \varepsilon\Phi_2(+\infty)^t + \varepsilon\Phi_1(+\infty)^t\right) \\
&\quad - \frac{1}{\nu_n^2}\Phi_1\big(\tilde{\xi}^{(n)}\big)\hat{G}_{11}\left(\int_0^{\tilde{\eta}^{(n)}} \Phi_1(s)^t ds - \varepsilon\Phi_1(+\infty)^t + \varepsilon\Phi_2(+\infty)^t\right) \\
&\quad + \mathcal{O}(\xi^{\frac{\alpha}{2}} n^{-\tau}) \\
&= K_J(\xi, \eta) - \frac{1}{4}\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}}\int_{\sqrt{\eta}}^{\infty}\left(J_{\alpha+1}(s) - \frac{2\alpha}{s}J_\alpha(s)\right) ds + \mathcal{O}(\xi^{\frac{\alpha}{2}} n^{-\tau}).
\end{aligned} \tag{6.26}$$

THE (1, 2)-ENTRY: Equation (2.53) gives

$$\left(-\frac{\partial}{\partial y}S_{n,1}\right)(x, y) = -\frac{\partial}{\partial y}K_n(x, y) + \Phi_1(x)A_{12}\Phi_2(y)^t + \Phi_1(x)\hat{G}_{11}\Phi_1(y)^t.$$

As before we then obtain from Propositions 6.1 and 6.4,

$$\begin{aligned}
&\frac{1}{\nu_n^2}\left[K_{n,1}^{(\nu_n)}\big(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\big)\right]_{12} = \frac{1}{\nu_n^4}\left(-\frac{\partial S_{n,1}}{\partial y}\right)\big(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\big) \\
&= -\frac{\partial}{\partial \eta}\left(\frac{1}{\nu_n^2}K_n\big(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}\big)\right) + \frac{1}{\nu_n^4}\Phi_1\big(\tilde{\xi}^{(n)}\big)A_{12}\Phi_2\big(\tilde{\eta}^{(n)}\big)^t + \frac{1}{\nu_n^4}\Phi_1\big(\tilde{\xi}^{(n)}\big)\hat{G}_{11}\Phi_1\big(\tilde{\eta}^{(n)}\big)^t \\
&= -\frac{\partial}{\partial \eta}K_J(\xi, \eta) - \frac{1}{8}\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}}\left(\frac{J_{\alpha+1}(\sqrt{\eta})}{\sqrt{\eta}} - \frac{2\alpha}{\eta}J_\alpha(\sqrt{\eta})\right) + \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}}\eta^{\frac{\alpha}{2}-1}}{n}\right).
\end{aligned} \tag{6.27}$$

THE (2, 1)-ENTRY: As for the (1, 1)-entry we first derive

$$\int_{\tilde{\xi}^{(n)}}^{\tilde{\eta}^{(n)}} \Phi_1(s)\, ds\, A_{12}\hat{C}_{22}^{-1}\left[\mathcal{O}(n^{-\tau})\varepsilon\Phi_1(+\infty)^t + \mathcal{O}(n^{-\tau})\varepsilon\Phi_2(+\infty)^t\right] = \mathcal{O}(n^{-\tau}),$$

using (6.19) instead of (6.15). With $(\varepsilon S_{n,1})(x, y) = -\int_x^y S_{n,1}(s, y)\, ds$ (see Proposition 2.1) we obtain from (2.53), (6.11), (6.19) and Proposition 6.6, in the same way as before,

$$\begin{aligned}
\frac{1}{\nu_n^2}\big[K_{n,1}^{(\nu_n)}\big(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\big)\big]_{21} &= (\varepsilon S_{n,1})(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}) - \frac{1}{2}\operatorname{sgn}(\xi-\eta)\\
&= -\int_{\xi}^{\eta}\frac{1}{\nu_n^2}K_n(\tilde{s}^{(n)},\tilde{\eta}^{(n)})\,ds\\
&\quad+\int_{\tilde{\xi}^{(n)}}^{\tilde{\eta}^{(n)}}\Phi_1(s)\,ds\,A_{12}\left(\int_0^{\tilde{\eta}^{(n)}}\Phi_2(s)^t ds-\varepsilon\Phi_2(+\infty)^t+\varepsilon\Phi_1(+\infty)^t\right)\\
&\quad+\int_{\tilde{\xi}^{(n)}}^{\tilde{\eta}^{(n)}}\Phi_1(s)\,ds\,\hat{G}_{11}\left(\int_0^{\tilde{\eta}^{(n)}}\Phi_1(s)^t ds-\varepsilon\Phi_1(+\infty)^t+\varepsilon\Phi_2(+\infty)^t\right)\\
&\quad-\frac{1}{2}\operatorname{sgn}(\xi-\eta)+\mathcal{O}(n^{-\tau})\\
&= -\int_{\xi}^{\eta}K_J(s,\eta)\,ds+\frac{1}{2}\int_{\sqrt{\xi}}^{\sqrt{\eta}}J_{\alpha+1}(s)\,ds\int_{\sqrt{\eta}}^{\infty}\left(J_{\alpha+1}(s)-\frac{2\alpha}{s}J_\alpha(s)\right)ds\\
&\quad-\frac{1}{2}\operatorname{sgn}(\xi-\eta)+\mathcal{O}(n^{-\tau}).
\end{aligned} \tag{6.28}$$

This completes the proof of Theorem 1.1. □

Proof of Corollary 1.2(b):

The case $\beta = 2$. This result can already be found in Ref. 23, see also.[12] Nevertheless we follow [Ref. 8, Sec 2.2] and present a somewhat different argument which is also useful for orthogonal and symplectic ensembles.

Using the representation of gap probabilities by Fredholm determinants, we have the following expression for the distribution of the smallest eigenvalue $\lambda_1(M)$,

$$\mathbb{P}_{n,2}\left(\lambda_1(M)\le\frac{s}{\nu_n^2}\right)=1-\det\left(I-\hat{K}_{n,2}|_{L^2((0,s])}\right), \tag{6.29}$$

where $\hat{K}_{n,2}$ denotes the integral operator with kernel

$$\hat{K}_{n,2}(\xi,\eta)=\frac{1}{\nu_n^2}K_n\big(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\big).$$

We now prove that (6.29) converges to $1-\det\big(I-K_J|_{L^2((0,s])}\big)$. As the trace class determinant is continuous with respect to the trace class norm it suffices to prove that

$$\Delta_n := \hat{K}_{n,2}-K_J$$

converges to zero in trace class norm when considered as an integral operator on $L^2((0, s])$. Denoting $H_n := H_{n,1} + H_{n,2} + H_{n,3}$ we obtain from (6.5), (6.6) that

$$\Delta_n(\xi, \eta) = \xi^{\frac{\alpha}{2}} \eta^{\frac{\alpha}{2}} H_n(\xi, \eta) \quad \text{and} \quad \frac{\partial^{k+j}}{\partial \xi^k \partial \eta^j} H_n(\xi, \eta) = \mathcal{O}(1/n)$$

for ξ, η in bounded subsets of $(0, \infty)$. Following[8] we formally write Δ_n as a product of two integral operators

$$\Delta_n = F_1 \cdot F_2 \equiv \left(\xi^{-\varepsilon} \frac{1}{D+I} \right) \cdot ((D+I)\xi^{\frac{\alpha}{2}+\varepsilon} \eta^{\frac{\alpha}{2}} H_n), \tag{6.30}$$

where $\varepsilon \in \mathbb{R}$ and D denotes differentiation. We may think of $\frac{1}{D+I}$ as shorthand for the integral operator

$$\left(\frac{1}{D+I} f \right)(\xi) := \int_0^\xi e^{\eta - \xi} f(\eta) \, d\eta \,. \tag{6.31}$$

Indeed, integration by parts then yields

$$\frac{1}{D+I}(f' + f) = f \quad \text{for all } f \in C^1(\mathbb{R}_+) \cap C^0([0, \infty)) \text{ with } f(0) = 0 \,.$$

Thus decomposition (6.30) with the interpretation of (6.31) is valid whenever $\frac{\alpha}{2} + \varepsilon > 0$. F_1 and F_2 can then be written as integral operators with kernels

$$F_1(\xi, \eta) = \xi^{-\varepsilon} e^{\eta - \xi} \mathbf{1}_{\{\eta < \xi\}} \,,$$

$$F_2(\xi, \eta) = \xi^{\frac{\alpha}{2}+\varepsilon-1} \eta^{\frac{\alpha}{2}} \left(\frac{\alpha}{2} + \varepsilon + \xi + \xi \frac{\partial}{\partial \xi} \right) H_n(\xi, \eta) = \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}+\varepsilon-1} \eta^{\frac{\alpha}{2}}}{n} \right),$$

uniformly for $\xi, \eta \in (0, s]$. Assuming in addition that $\frac{1-\alpha}{2} < \varepsilon < \frac{1}{2}$ we see that F_1 and F_2 are both Hilbert–Schmidt operators on $L^2((0, s])$, because their respective kernels lie in $L^2((0, s] \times (0, s])$. Moreover, $\|F_2\|_{HS} = \mathcal{O}(1/n)$ which in turn implies $\|\Delta_n\|_1 = \mathcal{O}(1/n)$, where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm and $\|\cdot\|_1$ denotes the trace norm for operators acting on $L^2((0, s])$. This completes the proof for unitary ensembles.

The case $\beta = 4$. A slight modification of the derivation in [Ref. 22, Sec. 8], which is described in [Ref. 8, 2.2.3], provides the following representation for the distribution of the smallest eigenvalue $\lambda_1(M)$,

$$\mathbb{P}_{\frac{n}{2},4}\left(\lambda_1(M) \leq \frac{s}{\nu_n^2} \right) = 1 - \sqrt{\det\left(I - \hat{K}_{\frac{n}{2},4}|_{L^2((0,s])^2} \right)} \,, \tag{6.32}$$

where $\hat{K}_{\frac{n}{2},4}$ denotes the integral operator with kernel

$$\hat{K}_{\frac{n}{2},4}(\xi, \eta) = \frac{1}{\nu_n^2} g(\xi) K_{\frac{n}{2},4}^{(\nu_n)}(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}) g(\eta)^{-1}, \quad g(\xi) = \begin{pmatrix} \xi^\delta & 0 \\ 0 & \xi^{-\delta} \end{pmatrix}$$

For the derivation of (6.32) one needs to ensure that both $\xi^{-\delta}\sqrt{w(\xi)}$ and $\xi^{\delta}\frac{d}{d\xi}\sqrt{w(\xi)}$ belong to $L^2((0,s])$. These conditions are satisfied if $1-\alpha < 2\delta < 1+\alpha$. From considerations which will become clear below we further restrict the choice of δ. From now on we assume that δ is a fixed number with $\max(0, \frac{1-\alpha}{2}) < \delta < \frac{1}{2}$. Our goal is to prove that (6.32) converges as $n \to \infty$ (n even) to

$$1 - \sqrt{\det\left(I - g(\xi)K^{(4)}(\xi,\eta)g(\eta)^{-1}|_{L^2((0,s])^2}\right)}\,.$$

Using again the continuity of the trace class determinant with respect to trace class norm it suffices to prove that each entry of

$$\Delta_n(\xi,\eta) := g(\xi)\left(\frac{1}{\nu_n^2}K^{(\nu_n)}_{\frac{n}{2},4}(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}) - K^{(4)}(\xi,\eta)\right)g(\eta)^{-1}$$

converges to zero in trace class norm when considered as an integral operator on $L^2((0,s])$. As in Ref. 8 we split $\Delta_n = \Delta_n^{(1)} + \Delta_n^{(2)}$, where the first term refers to the Christoffel–Darboux part and the latter corresponds to the correction term. For example, for the 11-entry we have

$$2\left[\Delta_n^{(1)}(\xi,\eta)\right]_{11} = \xi^{\delta}\eta^{-\delta}\left[\frac{1}{\nu_n^2}K_n\left(\tilde{\xi}^{(n)},\tilde{\eta}^{(n)}\right) - K_J(\xi,\eta)\right],$$

$$\begin{aligned}2\left[\Delta_n^{(2)}(\xi,\eta)\right]_{11} = \xi^{\delta}\eta^{-\delta}\Bigg[&-\frac{1}{\nu_n^2}\Phi_2(\tilde{\xi}^{(n)})A_{21}\int_0^{\tilde{\eta}^{(n)}}\Phi_1(s)^t\,ds\\ &-\frac{1}{\nu_n^2}\Phi_2(\tilde{\xi}^{(n)})G_{11}\int_0^{\tilde{\eta}^{(n)}}\Phi_2(s)^t\,ds\\ &-\frac{1}{4}\left(\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}} - \frac{2\alpha}{\xi}J_\alpha(\sqrt{\xi})\right)\int_0^{\sqrt{\eta}}J_{\alpha+1}(s)\,ds\Bigg].\end{aligned}$$

Since $0 < \delta < \frac{1}{2}$ one can prove the trace norm convergence $[\Delta_n^{(1)}]_{11} \to 0$ in exactly the same way as $\Delta_n \to 0$ was proven in the case $\beta = 2$. In order to treat $[\Delta_n^{(2)}]_{11}$ we first observe that the rank of this operator is bounded by $m+1$ for all n. We may therefore estimate the trace norm by the Hilbert–Schmidt norm (cf. [Ref. 8, (2.7)]) $\|[\Delta_n^{(2)}]_{11}\|_1 \leq \sqrt{m+1}\|[\Delta_n^{(2)}]_{11}\|_{HS}$. The above proof of part (ii) of Theorem 1.1 (see (6.22) and above) shows

$$\left[\Delta_n^{(2)}(\xi,\eta)\right]_{11} = \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}-1+\delta}\eta^{\frac{\alpha}{2}-\delta}}{n}\right).$$

This implies $\|[\Delta_n^{(2)}]_{11}\|_{HS} = \mathcal{O}(1/n)$, because both exponents $\frac{\alpha}{2} - 1 + \delta$ and $\frac{\alpha}{2} - \delta$ are larger than $-\frac{1}{2}$ by the choice of δ. This completes the proof that $[\Delta_n]_{11}$

converges to zero in trace norm, and also proves the corresponding result for $[\Delta_n]_{22}$, because $[\Delta_n]_{22}$ is the adjoint of the operator $[\Delta_n]_{11}$ acting on $L^2((0,s])$.

Applying the same method of proof to the 12-entry we obtain $[\Delta_n]_{12} = [\Delta_n^{(1)}]_{12} + [\Delta_n^{(2)}]_{12}$ where the correction part satisfies $[\Delta_n^{(2)}(\xi,\eta)]_{12} = \mathcal{O}(\frac{\xi^{\frac{\alpha}{2}-1+\delta}\eta^{\frac{\alpha}{2}-1+\delta}}{n})$ by (6.23) and $2[\Delta_n^{(1)}]_{12} = F_1 \cdot F_2$ can be written as a composition of integral operators with kernels

$$F_1(\xi,\eta) = -\xi^{\delta-\varepsilon} e^{\eta-\xi} \mathbf{1}_{\{\eta<\xi\}},$$
$$F_2(\xi,\eta) = \xi^{\frac{\alpha}{2}+\varepsilon-1}\eta^{\frac{\alpha}{2}+\delta-1}\left(\frac{\alpha}{2}+\varepsilon+\xi+\xi\frac{\partial}{\partial\xi}\right)\left(\frac{\alpha}{2}+\eta\frac{\partial}{\partial\eta}\right)H_n(\xi,\eta)$$
$$= \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}+\varepsilon-1}\eta^{\frac{\alpha}{2}+\delta-1}}{n}\right).$$

Choosing $\frac{1-\alpha}{2} < \varepsilon < \frac{1}{2}$ we ensure that F_1, F_2, $[\Delta_n^{(2)}]_{12}$ are Hilbert–Schmidt with $\|F_2\|_{HS} = \mathcal{O}(1/n)$ and $\|[\Delta_n^{(2)}]_{12}\|_{HS} = \mathcal{O}(1/n)$. As the rank of $[\Delta_n^{(2)}]_{12}$ is bounded above by $m+1$ we have proven the trace class convergence of $[\Delta_n]_{12}$ to 0.

Finally we turn to the 21-entry. From (6.25) we learn $[\Delta_n^{(2)}]_{21} = \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}-\delta}\eta^{\frac{\alpha}{2}-\delta}}{n}\right)$ and $2[\Delta_n^{(1)}]_{21} = F_1 \cdot F_2$ with kernels

$$F_1(\xi,\eta) = \xi^{-\delta} e^{\eta-\xi} \mathbf{1}_{\{\eta<\xi\}},$$
$$F_2(\xi,\eta) = \left(\xi^{\frac{\alpha}{2}} H_n(\xi,\eta) + \int_0^{\xi} t^{\frac{\alpha}{2}} H_n(t,\eta)dt\right)\eta^{\frac{\alpha}{2}-\delta} = \mathcal{O}\left(\frac{\eta^{\frac{\alpha}{2}-\delta}}{n}\right).$$

The choice of δ ensures that F_1, F_2, $[\Delta_n^{(2)}]_{21}$ are Hilbert–Schmidt with $\|F_2\|_{HS} = \mathcal{O}(1/n)$, $\|[\Delta_n^{(2)}]_{21}\|_{HS} = \mathcal{O}(1/n)$ and rank of $[\Delta_n^{(2)}]_{21} \le m+1$. This completes the proof for the symplectic case.

The case $\beta = 1$. We choose $\max(0, \frac{1-\alpha}{2}) < \delta < \frac{1}{2}$ and $g(\xi) = \begin{pmatrix} \xi^{\delta} & 0 \\ 0 & \xi^{-\delta} \end{pmatrix}$ as above. Following [Ref. 22, Sec. 9], [Ref. 8, Sec. 2.2.3] we may express the distribution of the smallest eigenvalue $\lambda_1(M)$ for even values of n by

$$\mathbb{P}_{n,1}\left(\lambda_1(M) \le \frac{s}{\nu_n^2}\right) = 1 - \sqrt{\det{}_2\left(I - \hat{K}_{n,1}|_{L^2((0,s])^2}\right)}, \qquad (6.33)$$

where

$$\hat{K}_{n,1}(\xi,\eta) = \frac{1}{\nu_n^2} g(\xi) K_{n,1}^{(\nu_n)}(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}) g(\eta)^{-1},$$

and the regularized 2-determinant $\det_2$ is defined by $\det_2(I+A) \equiv \det\left((I+A)e^{-A}\right) e^{tr(A_{11}+A_{22})}$ for 2×2 block operators $A = (A_{ij})_{i,j=1,2}$ with A_{11}, A_{22} in trace class and A_{12}, A_{21} Hilbert–Schmidt (cf. [Ref. 8, below Corollary 1.2], [(19)]). Define

$$\Delta_n(\xi,\eta) := g(\xi)\left(\frac{1}{\nu_n^2} K_{n,1}^{(\nu_n)}(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)}) - K^{(1)}(\xi,\eta)\right) g(\eta)^{-1}.$$

In order to prove the convergence of (6.33) to

$$1 - \sqrt{\det_2\left(I - g(\xi)K^{(1)}(\xi,\eta)g(\eta)^{-1}|_{L^2((0,s])^2}\right)},$$

it suffices to show that the diagonal blocks $[\Delta_n]_{11}$, $[\Delta_n]_{22}$ converge to zero in trace class and that the off-diagonals $[\Delta_n]_{12}$, $[\Delta_n]_{21}$ converge to zero in Hilbert–Schmidt norm. The convergence of the diagonal blocks is proven in exactly the same way as in the case $\beta = 4$. For the off-diagonals we learn from Theorem 1.1(iii) that

$$[\Delta_n(\xi,\eta)]_{12} = \mathcal{O}\left(\frac{\xi^{\frac{\alpha}{2}+\delta}\eta^{\frac{\alpha}{2}-1+\delta}}{n^\tau}\right), \qquad [\Delta_n(\xi,\eta)]_{21} = \mathcal{O}\left(\frac{\xi^{-\delta}\eta^{-\delta}}{n^\tau}\right).$$

The choice of δ ensures $\|[\Delta_n]_{12}\|_{HS} = \mathcal{O}(1/n)$ and $\|[\Delta_n]_{21}\|_{HS} = \mathcal{O}(1/n)$, completing the proof for orthogonal ensembles. Statement (b) of Corollary 1.1 is now established. □

6.2. The Soft Edge of the Spectrum

The proof of Theorem 1.4 is similar to the proofs of Theorem 1.1 and [Ref. 8, Theorem 1.1]. Instead of the property $\mathbf{e}A_{21}\mathbf{e}^t = -\frac{1}{2}\frac{n}{\beta_n}$, which was used to prove universality at the hard edge, we will need at the soft edge the following (quite remarkable) fact.

Proposition 6.7. *Let* $\mathbf{a}$ *be the* m*-dimensional row vector given by (6.14). As* $n \to \infty$,

$$\mathbf{a}A_{21}\mathbf{a}^t = \mathbf{a}A_{12}\mathbf{a}^t = -\frac{n}{\beta_n}\left(\frac{m}{2} + \mathcal{O}(n^{-1/m})\right). \tag{6.34}$$

Proof: Since $A_{12} = A_{21}^t$, see (2.14), we have $\mathbf{a}A_{21}\mathbf{a}^t = \mathbf{a}A_{12}\mathbf{a}^t$. Further, from Lemma 2.5 we have,

$$\mathbf{a}A_{21}\mathbf{a}^t = -\frac{n}{\beta_n}\left(\mathbf{a}Y\mathbf{a}^t + \mathcal{O}(n^{-1/m})\right), \quad \text{where } Y = \begin{pmatrix} Q & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Here, Q is the $(m-1) \times (m-1)$-matrix with entries $Q(i, j) = c_{i+j-1}$, where c_ℓ is given by (2.22). With the notation $d_k = \sum_{j=k+1}^{m-1} c_j$ as in the beginning of Section 3.3, we obtain from (6.14) and Proposition 3.3,

$$\mathbf{a}Y\mathbf{a}^t = \sum_{k=0}^{m-1} d_k + \frac{1}{2}\frac{m}{2m-1} = \frac{m}{2}.$$

This proves the Proposition. □

Furthermore, instead of Propositions 6.1–6.6 we will need the following two Propositions.

Proposition 6.8. *(cf. [Ref. 8, (3.8) and (3.56)]) There exists $c > 0$ such that, uniformly for $\xi, \eta \in [L_0, \infty)$, as $n \to \infty$*

$$\frac{\partial^{k+j}}{\partial\xi^k\partial\eta^j}\left[\frac{1}{\lambda_n^2}K_n\left(\xi^{(n)}, \eta^{(n)}\right)\right] = \frac{\partial^{k+j}}{\partial\xi^k\partial\eta^j}K_{\mathrm{Ai}}(\xi, \eta) + \mathcal{O}(n^{-1/3})e^{-c\xi}e^{-c\eta}, \quad (6.35)$$

$$\int_\xi^\infty \frac{1}{\lambda_n^2}K_n\left(s^{(n)}, \eta^{(n)}\right)ds = \int_\xi^\infty K_{\mathrm{Ai}}(s, \eta)\,ds + \mathcal{O}(n^{-1/3})e^{-c\xi}e^{-c\eta}, \quad (6.36)$$

$$\int_\xi^\eta \frac{1}{\lambda_n^2}K_n\left(s^{(n)}, \eta^{(n)}\right)ds = \int_\xi^\eta K_{\mathrm{Ai}}(s, \eta)\,ds + \mathcal{O}(n^{-1/3})e^{-c\min(\xi,\eta)}e^{-c\eta}. \quad (6.37)$$

Proof: The proof of (6.35) can be given by either following the path of the proof of [Ref. 8, (3.8)] or by adjusting the arguments of the proof of Proposition 6.1 making efficient use of the formulae presented in Ref. 23. Estimates (6.36) and (6.37) are immediate from (6.35) with $k = j = 0$. □

Proposition 6.9. *(cf. [Ref. 8, Proposition 4.1]) Let $j = 1, 2$. There exists $\tau > 0$ and $c > 0$ such that, uniformly for $\xi \in [L_0, \infty)$, as $n \to \infty$,*

$$\frac{1}{\lambda_n^2}\Phi_j\big(\xi^{(n)}\big) = \frac{1}{\sqrt{m}}\sqrt{\frac{\beta_n}{n}}\left[\mathrm{Ai}\,(\xi)\cdot\mathbf{a} + \mathcal{O}\left(\frac{e^{-c\xi}}{n^\tau}\right)\right], \tag{6.38}$$

$$\int_{\xi^{(n)}}^{\eta^{(n)}}\Phi_j(s)ds = \frac{1}{\sqrt{m}}\sqrt{\frac{\beta_n}{n}}\left[\int_\xi^\eta \mathrm{Ai}\,(s)ds\cdot\mathbf{a} + \mathcal{O}\left(\frac{e^{-c\min(\xi,\eta)}}{n^\tau}\right)\right], \tag{6.39}$$

$$\int_{\xi^{(n)}}^{\infty}\Phi_j(s)ds = \frac{1}{\sqrt{m}}\sqrt{\frac{\beta_n}{n}}\left[\int_\xi^\infty \mathrm{Ai}\,(s)ds\cdot\mathbf{a} + \mathcal{O}\left(\frac{e^{-c\xi}}{n^\tau}\right)\right], \tag{6.40}$$

$$\int_{\xi^{(n)}}^{\infty}\Phi_j(s)ds - \varepsilon\Phi_1(+\infty) - \varepsilon\Phi_2(+\infty)$$
$$= -\frac{1}{\sqrt{m}}\sqrt{\frac{\beta_n}{n}}\left[\int_{-\infty}^\xi \mathrm{Ai}\,(s)ds\cdot\mathbf{a} + \mathcal{O}(n^{-\tau})\right]. \tag{6.41}$$

Proof: Using Lemmas 4.8, 4.10 and 4.12, the proof of (6.38) is similar to the proof of [Ref. 8, (4.4)]. Estimate (6.40) is immediate from (6.38), and estimate (6.41) follows from (6.40), Proposition 6.3 and the fact that $\int_{-\infty}^{\infty}\mathrm{Ai}\,(s)ds = 1$. □

We have now the necessary ingredients to prove our Theorem for the soft edge.

Proof of Theorem 1.4: (i) The result for the $\beta = 2$ case is proven in Ref. 23 and follows also from (6.35) with $k = j = 0$.

(ii) The proof of the second part of the theorem (the case $\beta = 4$) is similar to the proofs of Theorem 1.1(ii) and [Ref. 8, Theorem 1.1: case $\beta = 4$].

THE (1, 1)- AND (2, 2)-ENTRY: By (2.52), (1.10) and (1.11) we have

$$\begin{aligned}\frac{2}{\lambda_n^2}\big[K_{\frac{n}{2},4}^{(\lambda_n)}\big(\xi^{(n)},\eta^{(n)}\big)\big]_{11} &= \frac{1}{\lambda_n^2}S_{\frac{n}{2},4}\big(\xi^{(n)},\eta^{(n)}\big)\\ &= \frac{1}{\lambda_n^2}K_n\big(\xi^{(n)},\eta^{(n)}\big) + \frac{1}{\lambda_n^2}\Phi_2\big(\xi^{(n)}\big)A_{21}\int_{\eta^{(n)}}^\infty \Phi_1(s)^t ds\\ &\quad + \frac{1}{\lambda_n^2}\Phi_2\big(\xi^{(n)}\big)G_{11}\int_{\eta^{(n)}}^\infty \Phi_2(s)^t ds.\end{aligned}$$

The asymptotics of the first term on the right hand side of the latter equation have been determined in part (i). From (6.38), (6.40), Proposition 6.7 and the facts that $A_{21} = \mathcal{O}(\frac{n}{\beta_n})$, $|\mathrm{Ai}\,(\xi)| \le Ce^{-\xi}$ and $|\int_\eta^\infty Ai(s)ds| \le Ce^{-\eta}$ for $\xi, \eta \in [L_0, \infty)$ and

$C > 0$ some constant, we have

$$\frac{1}{\lambda_n^2}\Phi_2\big(\xi^{(n)}\big)A_{21}\int_{\eta^{(n)}}^{\infty}\Phi_1(s)^t ds$$
$$= \left[\mathrm{Ai}(\xi)\cdot \mathbf{a} + \mathcal{O}\left(\frac{e^{-c\xi}}{n^\tau}\right)\right]\frac{1}{m}\frac{\beta_n}{n}A_{21}\left[\int_\eta^\infty \mathrm{Ai}(s)ds\cdot \mathbf{a}^t + \mathcal{O}\left(\frac{e^{-c\eta}}{n^\tau}\right)\right]$$
$$= -\frac{1}{2}\mathrm{Ai}(\xi)\int_\eta^\infty \mathrm{Ai}(s)ds + \mathcal{O}(n^{-\tau})e^{-c\xi}e^{-c\eta}.$$

Since G_{11} is skew symmetric, see Lemma 2.10, we have $\mathbf{a}G_{11}\mathbf{a}^t = 0$. Using in addition (6.38), (6.40) and the facts that $G_{11} = \mathcal{O}(\frac{n}{\beta_n})$ (see Corollary 2.13), $|\mathrm{Ai}(\xi)| \leq Ce^{-\xi}$ and $|\int_\eta^\infty Ai(s)ds| \leq Ce^{-\eta}$ for $\xi, \eta \in [L_0, \infty)$, we have,

$$\frac{1}{\lambda_n^2}\Phi_2\big(\xi^{(n)}\big)G_{11}\int_{\eta^{(n)}}^{\infty}\Phi_2(s)^t ds = \mathcal{O}(n^{-\tau})e^{-c\xi}e^{-c\eta}.$$

We conclude that,

$$\frac{2}{\lambda_n^2}\big[K^{(\lambda_n)}_{\frac{n}{2},4}\big(\xi^{(n)}, \eta^{(n)}\big)\big]_{11} = K_{\mathrm{Ai}}(\xi,\eta) - \frac{1}{2}\mathrm{Ai}(\xi)\int_\eta^\infty \mathrm{Ai}(s)ds + \mathcal{O}(n^{-\tau})e^{-c\xi}e^{-c\eta}. \tag{6.42}$$

THE (1, 2)-ENTRY: We conclude from (2.52) that

$$\left(-\frac{\partial}{\partial y}S_{\frac{n}{2},4}\right)(x,y) = -\frac{\partial}{\partial y}K_n(x,y) + \Phi_2(x)A_{21}\Phi_1(y)^t + \Phi_2(x)G_{11}\Phi_2(y)^t.$$

Using (1.10), (1.11), (6.35), (6.38) and Proposition 6.7, we obtain

$$\frac{2}{\lambda_n^2}\big[K^{(\lambda_n)}_{\frac{n}{2},4}\big(\xi^{(n)}, \eta^{(n)}\big)\big]_{12} = \frac{1}{\lambda_n^4}\left(-\frac{\partial}{\partial y}S_{\frac{n}{2},4}\right)\big(\xi^{(n)}, \eta^{(n)}\big)$$
$$= -\frac{\partial}{\partial\eta}\left(\frac{1}{\lambda_n^2}K_n\big(\xi^{(n)}, \eta^{(n)}\big)\right) + \frac{1}{\lambda_n^4}\Phi_2\big(\xi^{(n)}\big)A_{21}\Phi_1\big(\eta^{(n)}\big)^t + \frac{1}{\lambda_n^4}\Phi_2\big(\xi^{(n)}\big)G_{11}\Phi_2\big(\eta^{(n)}\big)^t$$
$$= -\frac{\partial}{\partial\eta}K_{\mathrm{Ai}}(\xi,\eta) - \frac{1}{2}\mathrm{Ai}(\xi)\mathrm{Ai}(\eta) + \mathcal{O}\left(\frac{e^{-c\xi}e^{-c\eta}}{n^\tau}\right). \tag{6.43}$$

THE (2, 1)-ENTRY: We employ $(\varepsilon S_{\frac{n}{2},4})(x,y) = -\int_x^\infty S_{\frac{n}{2},4}(s,y)ds$ of Proposition 2.1 and derive from (2.52) that

$$(\varepsilon S_{\frac{n}{2},4})(x,y) = -\int_x^\infty K_n(s,y)ds - \int_x^\infty \Phi_2(s)ds A_{21}\int_y^\infty \Phi_1(s)^t\, ds$$
$$- \int_x^\infty \Phi_2(s)ds G_{11}\int_y^\infty \Phi_2(s)^t\, ds. \tag{6.44}$$

As above, we obtain from (1.10), (1.11), (6.36), (6.40) and Proposition 6.7,

$$\frac{2}{\lambda_n^2}\big[K^{(\lambda_n)}_{\frac{n}{2},4}\big(\xi^{(n)},\eta^{(n)}\big)\big]_{21} = (\varepsilon S_{\frac{n}{2},4})\big(\xi^{(n)},\eta^{(n)}\big) = -\int_\xi^\infty \frac{1}{\lambda_n^2}K_n\big(s^{(n)},\eta^{(n)}\big)ds$$
$$-\int_{\xi^{(n)}}^\infty \Phi_2(s)ds\,A_{21}\int_{\eta^{(n)}}^\infty \Phi_1(s)^t\,ds - \int_{\tilde{\xi}^{(n)}}^\infty \Phi_2(s)\,ds\,G_{11}\int_{\tilde{\eta}^{(n)}}^\infty \Phi_2(s)^t ds$$
$$= -\int_\xi^\infty K_{\mathrm{Ai}}(s,\eta)ds + \frac{1}{2}\int_\xi^\infty \mathrm{Ai}\,(s)ds\int_\eta^\infty \mathrm{Ai}\,(s)ds + \mathcal{O}\left(n^{-\tau}\right)e^{-c\xi}e^{-c\eta}. \tag{6.45}$$

(iii) The proof of the third part of the theorem is similar to the proofs of Theorem 1.1(iii) and [Ref. 8, Theorem 1.1: case $\beta = 1$]. One starts with formula (2.54). Using (1.9), Proposition 2.1 together with Propositions 6.8, 6.9, and 6.7, the same arguments as described in the proof of 1.1(iii), prove the result. However, one needs to use some identities for Airy functions ([Ref. 8, (2.3)] and $\int_{-\infty}^\infty \mathrm{Ai}\,(s)\,ds = 1$) in order to convince oneself that

$$-\int_\xi^\eta K_{\mathrm{Ai}}(s,\eta)ds - \frac{1}{2}\int_\xi^\eta \mathrm{Ai}\,(s)\,ds\int_{-\infty}^\eta \mathrm{Ai}\,(s)\,ds$$
$$= -\int_\xi^\infty K_{\mathrm{Ai}}(s,\eta)ds - \frac{1}{2}\int_\xi^\eta \mathrm{Ai}\,(s)ds + \frac{1}{2}\int_\xi^\infty \mathrm{Ai}\,(s)ds\int_\eta^\infty \mathrm{Ai}\,(s)ds$$

which is needed to verify that the limit of the (2, 1)-entry agrees with the one stated in the theorem. □

6.3. Universality in the Bulk of the Spectrum

The proof of this theorem is similar to the proof of [Ref. 7, Theorem 1.1]. We need the following two Propositions.

Proposition 6.10. *Let $j = 1, 2$. As $n \to \infty$, uniformly for ξ, η in compact subsets of $\mathbb{R}$ and x in compact subsets of $(0, 1)$,*

$$\frac{1}{q_n^2}\Phi_j\left(\beta_n x + \frac{\xi}{q_n^2}\right) = \mathcal{O}\left(\frac{\sqrt{\beta_n}}{n}\right), \tag{6.46}$$

$$\varepsilon\Phi_j\left(\beta_n x + \frac{\xi}{q_n^2}\right) = \mathcal{O}\left(\sqrt{\frac{\beta_n}{n}}\right), \tag{6.47}$$

$$\int_{\beta_n x + \frac{\xi}{q_n^2}}^{\beta_n x + \frac{\eta}{q_n^2}} \Phi_j(s)ds = \mathcal{O}\left(\frac{\sqrt{\beta_n}}{n}\right). \tag{6.48}$$

Proof: Let $k \in \mathbb{Z}$. By (4.1), (1.19), Proposition 5.8 and Lemma 4.8(ii) we have, uniformly for ξ in compact subsets of $\mathbb{R}$ and x in compact subsets of (0, 1), as $n \to \infty$

$$\frac{1}{q_n^2}\phi_{n+k}\left[\beta_n x + \frac{\xi}{q_n^2}\right] = \frac{\beta_n}{n\omega_n(x)}\frac{1}{\sqrt{\beta_{n+k}}}\hat{\phi}_{n+k}\left[\frac{\beta_n}{\beta_{n+k}}\left(x + \frac{\xi}{n\omega_n(x)}\right)\right]$$
$$= \mathcal{O}\left(\frac{\sqrt{\beta_n}}{n}\right).$$

Further, with $j = 1, 2$, we have by (4.1), (1.19) and Lemma 4.11,

$$\frac{1}{q_n^2}\psi_j\left(\beta_n x + \frac{\xi}{q_n^2}\right) = \frac{\sqrt{\beta_n}}{n\omega_n(x)}\hat{\psi}_j\left(x + \frac{\xi}{n\omega_n(x)}\right) = \mathcal{O}\left(\frac{\sqrt{\beta_n}}{n}\right).$$

We now have proven (6.46). Similarly, (6.47) follows from (5.19) and (5.28). Finally (6.48) is immediate from (6.46). □

Proposition 6.11. *Uniformly for ξ, η in compact subsets of $\mathbb{R}$ and x in compact subsets of* (0, 1)*, as $n \to \infty$*

$$\frac{\partial^{k+j}}{\partial\xi^k\partial\eta^j}\left[\frac{1}{q_n^2}K_n\left(\beta_n x + \frac{\xi}{q_n^2}, \beta_n x + \frac{\eta}{q_n^2}\right)\right] = \frac{\partial^{k+j}}{\partial\xi^k\partial\eta^j}K_\infty(\xi - \eta) + \mathcal{O}\left(\frac{1}{n}\right), \tag{6.49}$$

$$-\int_{\beta_n x + \frac{\xi}{q_n^2}}^{\beta_n x + \frac{\eta}{q_n^2}} K_n\left(s, \beta_n x + \frac{\eta}{q_n^2}\right)ds = \int_0^{\xi-\eta} K_\infty(s)ds + \mathcal{O}\left(\frac{1}{n}\right). \tag{6.50}$$

Proof: It is straightforward to modify the proof of Proposition 6.1 to derive the desired result. □

Proof of Theorem 1.6: (i) The case $\beta = 2$ has been proven in [Ref. 23, Theorem 2.8(a)].

(ii) We only consider the case $\beta = 1$. The case $\beta = 4$ is proved in a completely analogous fashion.

THE (1, 1)- AND (2, 2)-ENTRY: Since, by (1.11) and (1.9),

$$\left[K_{n,1}^{(q_{n,1})}(x, y)\right]_{11} = S_{n,1}(x, y)$$

we obtain from (2.44), (1.20), (6.46), (6.47) and the fact that $A_{12} = \mathcal{O}(\frac{n}{\beta_n}) = \hat{G}_{11}$ (see Lemma 2.5 and Corollary 2.13) and $q_{n,1} = q_n$,

$$\begin{aligned}
&\frac{1}{q_{n,1}^2}\left[K_{n,1}^{(q_{n,1})}\left(\beta_n x + \frac{\xi}{q_{n,1}^2}, \beta_n x + \frac{\eta}{q_{n,1}^2}\right)\right]_{11} \\
&\quad = \frac{1}{q_n^2}K_n\left(\beta_n x + \frac{\xi}{q_n^2}, \beta_n x + \frac{\eta}{q_n^2}\right) + \mathcal{O}\left(\frac{\sqrt{\beta_n}}{n}\right)\mathcal{O}\left(\frac{n}{\beta_n}\right)\mathcal{O}\left(\sqrt{\frac{\beta_n}{n}}\right) \\
&\quad = K_\infty(\xi - \eta) + \mathcal{O}(n^{-1/2}).
\end{aligned} \tag{6.51}$$

THE (1, 2)-ENTRY: Since, by (1.11) and (1.9), $[K_{n,1}^{(q_{n,1})}(x, y)]_{12} = -\frac{1}{q_{n,1}^2}\frac{\partial}{\partial y}S_{n,1}(x, y)$, we obtain from (2.44), (6.49), (6.46) and the facts that $A_{12} = \mathcal{O}(\frac{n}{\beta_n}) = \hat{G}_{11}$ and $q_{n,1} = q_n$,

$$\begin{aligned}
&\frac{1}{q_{n,1}^2}[K_{n,1}^{(q_{n,1})}\left(\beta_n x + \frac{\xi}{q_{n,1}^2}, \beta_n x + \frac{\eta}{q_{n,1}^2}\right)]_{12} \\
&\quad = -\frac{\partial}{\partial \eta}\left[\frac{1}{q_n^2}K_n\left(\beta_n x + \frac{\xi}{q_n^2}, \beta_n x + \frac{\eta}{q_n^2}\right)\right] + \mathcal{O}\left(\frac{\sqrt{\beta_n}}{n}\right)\mathcal{O}\left(\frac{n}{\beta_n}\right)\mathcal{O}\left(\frac{\sqrt{\beta_n}}{n}\right) \\
&\quad = -\frac{\partial}{\partial \eta}K_\infty(\xi - \eta) + \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned} \tag{6.52}$$

Since $-\frac{\partial}{\partial \eta}K_\infty(\xi - \eta) = \frac{\partial}{\partial \xi}K_\infty(\xi - \eta)$, this proves the convergence of the (1, 2)-entry.

THE (2, 1)-ENTRY: We use the formula $(\varepsilon S_{n,1})(x, y) = -\int_x^y S_{n,1}(s, y)ds$ of Proposition 2.1 (in contrast to the edge cases, one should use the same formula also for $\beta = 4$) and arrive via (1.11) and (1.9) at

$$\begin{aligned}
\left[K_{n,1}^{(q_{n,1})}(x, y)\right]_{21} &= q_{n,1}^2\left[(\varepsilon S_{n,1})(x, y) - \frac{1}{2}\operatorname{sgn}(x - y)\right] \\
&= -q_{n,1}^2\left[\int_x^y S_{n,1}(s, y)ds + \frac{1}{2}\operatorname{sgn}(x - y)\right].
\end{aligned}$$

This together with (2.44), (6.50), (6.47), (6.48) and the facts that $A_{12} = \mathcal{O}(\frac{n}{\beta_n}) = \hat{G}_{11}$ and $q_{n,1} = q_n$ yields

$$\frac{1}{q_{n,1}^2}\left[K_{n,1}^{(q_{n,1})}\left(\beta_n x + \frac{\xi}{q_{n,1}^2}, \beta_n x + \frac{\eta}{q_{n,1}^2}\right)\right]_{21}$$

$$= -\int_{\beta_n x + \frac{\xi}{q_n^2}}^{\beta_n x + \frac{\eta}{q_n^2}} K_n\left(s, \beta_n x + \frac{\eta}{q_n^2}\right) ds - \frac{1}{2}\mathrm{sgn}(\xi - \eta) + \mathcal{O}\left(\frac{1}{n}\right)$$

$$= \int_0^{\xi-\eta} K_\infty(s) ds - \frac{1}{2}\mathrm{sgn}(\xi - \eta) + \mathcal{O}\left(\frac{1}{n}\right). \tag{6.53}$$

This completes the proof in the $\beta = 1$ case. □

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